Curvature
Just as we use lines as reference when discussing the slope of a function, we use circles as a reference when discussing the curvature of a function.

With slope, a large absolute value indicates a steep incline whereas a small absolute value indicates a shallow incline. The sign on the slope indicates whether the line is increasing or decreasing.

Since a circle neither increases nor decreases, there is no need for negative curvatures. It would be nice if we had a definition for curvature that leads to large curvature values for really curvy circles and small curvature values for circles that are not so curvy.

Really curvy circles have \textit{small radii}.

Not so curvy circles have \textit{large radii}.

The osculating circle to a function at a given point is the circle in the osculating plane that has the same curvature as the given function at that point, has the same tangent line as the function at that point, and lies on the same side of the tangent line as the function at that point.

The osculating circle for the function shown in Figure 1 has been drawn at the point: \((1,1)\). What is the curvature of the function at this point?

\[
\kappa = \frac{1}{\rho} = \frac{1}{\sqrt{2}}
\]

At the point \((4,-1)\), the curvature of the ellipse in Figure 2 is \(\frac{1}{4}\).

Let's find the center of the osculating circle and draw the circle onto the picture.

\[
\rho = \frac{1}{\kappa} = 4
\]

\[
\hat{r}(t_{0}) = \hat{r}(t_{0}) + \rho \hat{N}(t_{0})
\]

\[
= \left< 1, -1 \right> + 4 \left< -1, 0 \right>
\]

\[
= \left< 0, -1 \right>
\]
Let’s find the curvature of the ellipse in Figure 2 at the point \((3, 1)\). Let’s then determine the center of the osculating circle at that point and draw the circle onto the figure.

Equation for ellipse: \[\frac{(x-3)^2}{1^2} + \frac{(y+1)^2}{2^2} = 1\]

So: \[4\left(x-3\right)^2 + (y+1)^2 = 4\]

\[= \frac{d}{dx} \left(4\left(x-3\right)^2 + (y+1)^2\right) = \frac{d}{dx}(4)\]

\[= 8\left(x-3\right) \cdot \frac{d}{dx}(x-3) + 2(y+1) \cdot \frac{d}{dx}(y+1) = 0\]

\[= 8\cdot (x-3) \cdot 1 + 2(y+1) \cdot \frac{d}{dx}(y+1) = 0\]

\[= \frac{dy}{dx} = \frac{4x-12}{y+1}\]

\[\Rightarrow \frac{d}{dx} \left(\frac{dy}{dx}\right) = \frac{d}{dx} \left(- \frac{4x-12}{y+1}\right)\]

\[= \frac{d^2y}{dx^2} = \frac{-4(y+1) - (4x-12)\frac{dy}{dx}}{(y+1)^2}\]

\[= \frac{-4(y+1) - (4x-12)\frac{dy}{dx}}{(y+1)^2}\]

\[K_1(3, 1) = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}\]

\[= \frac{-4(y+1) - (4x-12)\frac{dy}{dx}}{(y+1)^2}\]

\[K_1(3, 1) = 2 = \rho(3, 1) = \frac{1}{2}\]

The center of the circle \((3, 1) + \frac{1}{2} \cdot \rho(3, 1)\) is \((\frac{7}{2}, \frac{1}{2})\).
The function \( y = f(x) = \frac{-3x - 8}{x + 4} \) is shown in Figure 3. Let’s find the center of the osculating circle at the point \((-2, -1)\), let’s draw the circle, and finally let’s state an implicit function that graphs to the osculating circle.

One formula for curvature is

\[
\kappa = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}
\]

\[
f'(x) = \frac{-3(x+4) - (-3x - 8)(1)}{(x+4)^2}
\]

\[
= -\frac{4}{(x+4)^2}
\]

\[
f''(x) = \frac{8}{(x+4)^3}
\]

\[
K \bigg|_{(-2, -1)} = K(-2) = \frac{1}{\left[1 + \left(\frac{8}{(x+4)^2}\right)\right]^{3/2}}
\]

\[
= \frac{1}{2^{3/2}}
\]

\[
\rho(-2) = \frac{1}{K(-2)} = 2^{3/2}
\]

The center lies at \((-2, -1) + \rho(-2) \hat{N}(-2)\)

\[
\hat{N}(-2) = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}
\]

\[
\rho(-2) \hat{N}(-2) = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \frac{2^{3/2}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

One equation of the osculating circle is

\[
x^2 + (y - 1)^2 = \frac{3}{2} \left( \frac{1}{\sqrt{2}} \right)^2
\]
Figure 4 shows a graph of the vector function \( \mathbf{r}(t) = \left\langle 1.5 + \cos(t), 3 + 2 \cos(t), 4 + 2 \sin(t) \right\rangle \) in its osculating plane, \( y = 2x \). Let’s find the center of the osculating circle to the curve when \( t = 0 \). Let’s also find a vector function that describes this osculating circle.

\[
\mathbf{r}'(t) = \left\langle -\sin(t), -2\sin(t), 2\cos(t) \right\rangle
\]
\[
\mathbf{r}''(t) = \left\langle -\cos(t), -2\cos(t), -2\sin(t) \right\rangle
\]
\[
\kappa(0) = \frac{1}{\|\mathbf{r}'(0)\|^{3}} \left\| \mathbf{r}'(0) \times \mathbf{r}''(0) \right\|
\]
\[
= \frac{1}{\sqrt{5}} \left\| \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \right\|
\]
\[
= \frac{\sqrt{5}}{4}
\]

The osculating circle lies on the plane \( y = 2x \) and on the sphere with radius \( \rho(0) = \frac{4}{\sqrt{5}} \) whose center is at the point \( \mathbf{r}(0) + \rho(0) \hat{\mathbf{n}}(0) \) with direction \( \left\langle -1, -2, 0 \right\rangle \).

\[
\text{Center: } \left\langle 2.5, 4, \frac{4}{\sqrt{5}} \right\rangle \\
= \left\langle 1.7, 2.4, 4 \right\rangle
\]

So, one equation for the sphere in Figure 5 is
\[
(x-1.7)^2 + (y-2.4)^2 + (z-4)^2 = 3.2
\]
\[
\rho(0)^2 = \left( \frac{4}{\sqrt{5}} \right)^2
\]
Planes
At a given point along a curve, the unit tangent, normal, and binormal vectors define three planes we associate with curve at that point.

Figure 6a: The osculating plane contains the tangent line and the normal line.
The osculating plane is perpendicular to the binormal line.

Figure 6b: The osculating plane is parallel to the unit tangent vector and the unit normal vector.
The osculating plane is perpendicular to the binormal vector.

Figure 7a: The rectifying plane contains the tangent line and the binormal line.
The rectifying plane is perpendicular to the normal line.

Figure 7b: The rectifying plane is parallel to the unit tangent vector and the binormal vector.
The rectifying plane is perpendicular to the unit normal vector.

Figure 8a: The normal plane contains the normal line and the binormal line.
The normal plane is perpendicular to the tangent line.

Figure 8b: The normal plane is parallel to the unit normal vector and the binormal vector.
The normal plane is perpendicular to the unit tangent vector.

NOTE: The length of each unit vector has been exaggerated for illustrative purposes.
Figure 9 shows the vector function \( \mathbf{r}(t) = \langle 4\sin(t) \cos(2t), 7, 3\sin(2t)\cos(t) \rangle \).

Let's intuit the equations of the osculating, normal, and rectifying planes where \( t = 0 \).

The osculating plane is \( y = 7 \)

\[
\begin{align*}
\frac{dx}{dt} &= 6 \cos(2t) \cos(t) + 3 \sin(2t) \sin(t) \\
\frac{dy}{dt} &= 4 \cos(t) \cos(2t) - 7 \sin(2t) \sin(t) \\
\frac{dz}{dt} &= 9 \sin(t) \cos(t) - 3 \sin(2t) \cos(t)
\end{align*}
\]

Using these derivatives, we find the equation of the osculating plane.

The normal plane is perpendicular to the tangent line in this picture; it's the plane \( \perp \) to \( y = 7 \) along the normal line \( z = -\frac{x}{3} \). Similarly, the rectifying is \( \perp \) to \( y = 7 \) along the tangent \( z = \frac{x}{3} \).

Let's verify the equation of the normal plane using "traditional" techniques.

A point on the plane is \( \mathbf{r}(0) = \langle 0, 7, 0 \rangle \)

A normal vector for the plane is \( \mathbf{n}(0) \)

\[
\mathbf{r}(t) = \langle 4\cos(t) \cos(2t), 7, 3\sin(2t)\cos(t) \rangle
\]

\[
\mathbf{n}(0) = \frac{\mathbf{r}'(0)}{\sqrt{\mathbf{r}'(0) \cdot \mathbf{r}'(0)}}
\]

\[
\Rightarrow \mathbf{n}(0) \perp \mathbf{n}(0) \Rightarrow \langle 2, 0, 3 \rangle \perp \mathbf{n}(0)
\]

\[
\langle 2, 0, 3 \rangle \cdot \langle x, y - 7, z - 0 \rangle = 0
\]

\[
\Rightarrow 2x + 3z = 0
\]

\[
\Rightarrow z = -\frac{2}{3} x \quad \text{or} \quad \mathbf{r}(0)
\]
Every point on the O.C. lies on the plane \( y = 2x \).

Let us use this to effect a substitution upon the sphere equation,

\[
\begin{align*}
(x-1.7)^2 + (2x-3.4)^2 + (z-4)^2 &= 3.2 \\
(x-1.7)^2 + 4(x-1.7)^2 + (z-4)^2 &= 3.2 \\
5(x-1.7)^2 + (z-4)^2 &= 3.2 \\
\frac{5(x-1.7)^2}{3.2} + \frac{(z-4)^2}{2} &= 1 \\
\frac{(x-1.7)^2}{0.64} + \frac{(z-4)^2}{3.2} &= 1
\end{align*}
\]

This \( xz \)-relationship can be maintained by letting \( x = 1.7 + 0.8 \cos(t) \) and \( z = 4 + \sqrt{2} \sin(t) \).

\[ y = 2x, \text{ ergo } y = 3.4 + 1.6 \cos(t) \]

A vector function that models the osculating circle is

\[ \vec{r}(t) = (1.7 + 0.8 \cos(t), 3.4 + 1.6 \cos(t), 4 + \sqrt{2} \sin(t)) \]