The first derivative of a vector-valued function

A graph of the vector-valued function \( \mathbf{r}(t) = \left( 2 + \cos \left( \frac{3}{2} t \right), 2 + \cos \left( \frac{3}{2} t \right) \right) \sin(t) \) is shown in Figure 1. Several values of \( \mathbf{r}(t) \) and \( \mathbf{r}'(t) \) are shown in Table 1.

Sketch each of the derivative vectors with their tails at the corresponding point on \( \mathbf{r}(t) \) and make some observations.

![Graph of \( \mathbf{r}(t) \)](image)

**Figure 1: \( \mathbf{r}(t) \)**

### Three properties of \( \mathbf{r}'(t_0) \) when \( \mathbf{r}'(t_0) \neq 0 \).
- \( \mathbf{r}'(t_0) \) is parallel to the tangent line to \( \mathbf{r} \) at \( t_0 \).
- \( \mathbf{r}'(t_0) \) points in the direction of motion (in the increasing \( t \)-values).
- \( |\mathbf{r}'(t_0)| \) is the speed at time \( t_0 \).

### Table 1: \( \mathbf{r}(t) \) and \( \mathbf{r}'(t) \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \mathbf{r}(t) )</th>
<th>( \mathbf{r}'(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( (3,0) )</td>
<td>( (0,3) )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( (0,1.3) )</td>
<td>( (-1.3,-1.1) )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( (-2,0) )</td>
<td>( (-1.5,-2) )</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>( (0,-2.7) )</td>
<td>( (2.7,1.1) )</td>
</tr>
</tbody>
</table>
Let's find the point on the curve \( \mathbf{r}(t) = \langle 2 \cos(t), 2 \sin(t), e^t \rangle \); \( t \in [0, \pi] \) where the tangent line to the curve is parallel to the plane \( \sqrt{3} x + y = 1 \). Let's also state a vector function that graphs to that line.

\[
\langle \sqrt{3}, 1, 0 \rangle \perp \text{plane} \quad \mathbf{r}'(t_0) \parallel \mathbf{v} + t \mathbf{w}
\]

\[
\mathbf{r}'(t_0) = \langle -2 \sin(t), 2 \cos(t), e^t \rangle
\]

\[
\mathbf{v}'(t_0) \parallel \text{plane} \quad \therefore \quad \mathbf{v}'(t_0) \perp \langle \sqrt{3}, 1, 0 \rangle
\]

Ergo \( \langle \sqrt{3}, 1, 0 \rangle \cdot \langle -2 \sin(t), 2 \cos(t), e^t \rangle = 0 \)

\[
-2 \sqrt{3} \sin(t) + 2 \cos(t) = 0
\]

\[
-2 \sqrt{3} \sin(t) = -2 \cos(t)
\]

\[
\tan(t) = \frac{1}{\sqrt{3}}
\]

\[
t = \frac{\pi}{6}
\]

The value at \( t_0 = \frac{\pi}{6} \)

\[
\therefore \quad \mathbf{r}(\pi/6) = \langle \sqrt{3}, 1, e^{\pi/6} \rangle
\]

\[
\mathbf{v}'(\pi/6) = \langle -1, \sqrt{3}, e^{\pi/6} \rangle
\]

\[
\therefore \quad \text{The line is described thus:}
\]

\[
Q(t) = \langle \sqrt{3}, 1, e^{\pi/6} \rangle + t \langle -1, \sqrt{3}, e^{\pi/6} \rangle
\]
Let's determine the point of intersection between the curves \( \vec{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle \) and \( \vec{r}_2(t) = \langle 3-t, t-2, t^2 \rangle \). Then let's determine the smallest angle formed by the tangent lines to the two curves at that point.

\[
\begin{align*}
\chi_1 &= \chi_2 : & t_1 &= 3 - t_2 \\
y_1 &= y_2 : & 1 - t_1 &= t_2 - 2 \\
z_1 &= z_2 : & 3 + t_1^2 &= t_2^2 \\
\end{align*}
\]

\[
\begin{align*}
\vec{r}_1(t) &= \langle t, 1-t, 3+t^2 \rangle \\
\vec{r}_2(t) &= \langle 3-t, t-2, t^2 \rangle \\
\end{align*}
\]

\[
\begin{align*}
t_1 &= 1 \\
t_2 &= 2 \\
\end{align*}
\]

\[
\vec{r}_1'(t) = \langle 1, -1, 2t \rangle \\
\vec{r}_2'(t) = \langle -1, 1, 2t \rangle
\]

\[
\begin{align*}
\Theta &= \cos^{-1} \left( \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} \right) \\
\end{align*}
\]

\[\approx 54.7^\circ\]
Let’s evaluate, by hand, \( \int_0^1 (4t \hat{j} - te^t \hat{k}) \, dt \):

\[
\int_0^1 (4t \hat{j} - te^t \hat{k}) \, dt = \left( \int_0^1 4t \, dt \right) \hat{j} - \left( \int_0^1 te^t \, dt \right) \hat{k}
\]

\[
= \left[ 2t^2 \right]_0^1 \hat{j} - \left[ t e^t \right]_0^1 \hat{k}
\]

\[
= 2 \hat{j} - (1 \cdot e - 0) \hat{k}
\]

\[
= 2 \hat{j} - \left( \frac{1}{2} e^0 - \frac{1}{2} e^0 \right) \hat{k}
\]

\[
< 0, 2, \frac{1}{2} e^0 - \frac{1}{2} e^0 >
\]

Let’s find the arc length and net displacement along the function \( \vec{r}(t) = (2 \cos(3t), \sin(4t)) \) between times \( t = 0 \) and \( t = \frac{\pi}{6} \). Let’s use our calculator to perform any necessary integration.

\[
\text{Distance travelled} = \int_0^{\pi/6} \sqrt{\left( \frac{d}{dt} \vec{r}(t) \right)^2} \, dt
\]

\[
= \int_0^{\pi/6} \sqrt{(-6 \sin(3t))^2 + (-4 \cos(4t))^2} \, dt
\]

\[
\approx 2.55
\]

Figure 3: Displacement

Figure 4: Arc length