Vectors, lines, planes, and general 3-d graphing ... some things I expect you know! :-O

Q1: What is a fast and easy way to check and see whether or not two vectors are perpendicular? What does “perpendicular” even mean in this context?

\[ \vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0 \quad (\text{and } \vec{u} \neq \vec{0} \text{ and } \vec{v} \neq \vec{0}) \]

If \( \vec{u} \cdot \vec{v} = 0 \), \( \vec{u} \) and \( \vec{v} \) are orthogonal.

Determine which pairs of the following vectors are perpendicular. What is the smallest angle formed by the other pair(s) when drawn tail-to-tail?

\[ \vec{u} = \langle 2, -4, 3 \rangle, \quad \vec{v} = \langle 1, 2, 2 \rangle, \quad \text{and} \quad \vec{w} = \langle 3, -12, -18 \rangle \]

\[ \vec{u} \cdot \vec{v} = (2)(1) + (-4)(2) + 3(2) = 0 \]

\[ \vec{u} \cdot \vec{w} = (2)(3) + (-4)(-12) + 3(-18) = 0 \]

\[ \vec{v} \cdot \vec{w} = (1)(3) + (2)(-12) + 2(-18) \neq 0 \]

Definition: \( \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos(\theta) \)

\[ \theta = \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|} \right) \approx 119^\circ \]

Q2: How can we find a vector that is perpendicular to two non-parallel vectors?

If \( \vec{u} \parallel \vec{v} \) \((\vec{u} \neq \vec{0}, \vec{v} \neq \vec{0})\), then

\[ \vec{u} \times \vec{v} \perp \vec{u} \text{ and } \vec{u} \times \vec{v} \perp \vec{v} \]

Find a vector perpendicular to both \( \vec{v} = \langle 1, 2, 2 \rangle \) and \( \vec{w} = \langle 3, -12, -18 \rangle \).

\[ \vec{v} \times \vec{w} = \langle -12, 24, -18 \rangle \parallel \langle 2, -4, 3 \rangle \]

Check:

\[ \vec{v} \cdot \langle 2, -4, 3 \rangle = 0 \quad \text{and} \quad 2\vec{v}, \langle -4, 3 \rangle \parallel \vec{w} \]

Speedy review of vectors, lines, and planes: Sections 9.1 – 9.5
**Q3:** What is a fast and easy way to check and see whether or not two vectors are parallel? What does “parallel” even mean in this context?

\[ \vec{u} \parallel \vec{v} \iff \vec{u} = k \vec{v} \text{ where } k \neq 0 \]

**Note:** \( \vec{u} \parallel \vec{v} \) (\( \vec{u} \neq \vec{0}, \vec{v} \neq \vec{0} \)) iff \( \vec{u} \times \vec{v} = \vec{0} \).

Parallel means that the vectors point in the same direction (\( \uparrow \uparrow \)) or in 180° opposition (\( \uparrow \downarrow \)).

What two vectors that we have already encountered must be parallel? Verify this relationship.

\[
\text{I gave this up on the last page } \vec{u} \text{ and } \vec{v} \times \vec{w}.
\]

**Q4:** What is a unit vector? How can we find a unit vector that is parallel to a given vector \( \vec{u} \)?

\( \hat{\vec{v}} \) is a unit vector iff \( |\hat{\vec{v}}| = 1 \).

To find \( \vec{w} \parallel \vec{v} \), we normalize \( \vec{v} \).

Find the unit vector that points in the opposite direction as the vector \( \vec{v} = 3\hat{i} - 8\hat{k} \)

\[
\hat{\vec{v}} = -\frac{\left< 3, 0, -8 \right>}{|\left< 3, 0, -8 \right>|} = -\frac{\left< 3, 0, -8 \right>}{\sqrt{3^2 + 0^2 + (-8)^2}} = -\frac{1}{\sqrt{73}} \left< 3, 0, -8 \right> = \left< -\frac{3}{\sqrt{73}}, 0, \frac{8}{\sqrt{73}} \right>
\]
Q5: What two objects enable you to find a vector equation for a line? How are they used?

We need a point, \((x_0, y_0, z_0) = \vec{P}\), and a direction vector \(<a, b, c> = \vec{v}\). Then the line can be described as:

\[
\vec{r}(t) = \vec{P}_0 + \vec{v}t
\]

\[
= <x_0+at, y_0+bt, z_0+ct>
\]

Note: \(\vec{r}(t) \parallel \vec{v}\)

When vectors are involved, parallel means same or opposite direction; it has nothing to do with intersection.

Q4: What two objects enable you to find an equation for a plane? How are they used?

We need a point \((x_0, y_0, z_0)\) and a normal vector \(\vec{n}\).

\((x_0, y_0, z_0)\) is a specific point on \(P\), \((x_1, y_1, z_1)\) is any point on \(P\),

\[
\forall (x, y, z) \in P, \text{ either } \vec{X}_0X_1 \perp \vec{n} \text{ or } X_0 = X_1 <X_0X_1, \vec{n}> = 0
\]

Ergo, the plane equation is \(<a, b, c>. <x-x_0, y-y_0, z-z_0> = 0\).
Find an equation of the plane containing the points \( A = (9, -2, 4) \), \( B = (7, 0, -4) \), and \( C = (3, 3, 2) \).

**Preliminary work**
1. Find a normal vector for the plane
2. Find a point on the plane

\[
\begin{align*}
\text{The equation of } \mathbb{P} & : \\
\langle 18, 22, 1 \rangle \cdot \langle x - 7, y + 4, z + 4 \rangle = 0 \\
\Rightarrow 18(x - 7) + 22(y + 4) + 1(z + 4) = 0 \\
\Rightarrow 18x + 22y + z = 122
\end{align*}
\]

Find a vector equation for the line segment connecting the points \( A = (9, -2, 4) \) and \( B = (7, 0, -4) \).

**Preliminary work**
1. Find a direction vector for the line
2. Find a point on the line

\[
\begin{align*}
\overrightarrow{AB} & = \langle -2, 2, -8 \rangle \\
\overrightarrow{AB} & \parallel \ell \\
A & \in \ell \\
\Rightarrow \quad \ell & \text{ is an equation for } \ell \\
\odot(t) & = \langle 9, -2, 4 \rangle + \langle -2, 2, -8 \rangle t
\end{align*}
\]