Vectors, lines, planes, and general 3-d graphing ... some things I expect you know! :-O

Q1: What is a fast and easy way to check and see whether or not two vectors are perpendicular? What does “perpendicular” even mean in this context?

Determine which pairs of the following vectors are perpendicular. What is the smallest angle formed by the other pair(s) when drawn tail-to-tail?

\[ \vec{u} = \langle 2, -4, 3 \rangle, \; \vec{v} = \langle 1, 2, 2 \rangle, \text{ and } \vec{w} = \langle 3, -12, -18 \rangle \]

Q2: How can we find a vector that is perpendicular to two non-parallel vectors?

Find a vector perpendicular to both \( \vec{v} = \langle 1, 2, 2 \rangle \) and \( \vec{w} = \langle 3, -12, -18 \rangle \).
Q3: What is a fast and easy way to check and see whether or not two vectors are parallel? What does “parallel” even mean in this context?

What two vectors that we have already encountered must be parallel? Verify this relationship.

Q4: What is a unit vector? How can we find a unit vector that is parallel to a given vector \( \vec{u} \)?

Find the unit vector that points in the opposite direction as the vector \( \vec{v} = 3\hat{i} - 8\hat{k} \).
Q5: What two objects enable you to find a vector equation for a line? How are they used?

Q4: What two objects enable you to find an equation for a plane? How are they used?
Find an equation of the plane containing the points \( A = (9, -2, 4) \), \( B = (7, 0, -4) \), and \( C = (3, 3, 2) \).

**Preliminary work**

1. Find a normal vector for the plane
2. Find a point on the plane

Find a vector equation for the line segment connecting the points \( A = (9, -2, 4) \) and \( B = (7, 0, -4) \).

**Preliminary work**

1. Find a direction vector for the line
2. Find a point on the line
Basic Vector Language and Vector Arithmetic

Fact 1: The three standard unit vectors are \( \hat{i} = \langle 1, 0, 0 \rangle \), \( \hat{j} = \langle 0, 1, 0 \rangle \) and \( \hat{k} = \langle 0, 0, 1 \rangle \). The vector \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) can also be written as \( \vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \). Consequently, the components of the vector \( \vec{u} \); \( u_1 \), \( u_2 \), and \( u_3 \); are called, respectively, the \( \hat{i} \) – component, the \( \hat{j} \) – component, and the \( \hat{k} \) – component.

Fact 2: To add or subtract two vectors you add or subtract the corresponding components of the two vectors. To multiply a vector by a scalar (number), you multiply each component of the vector by the scalar.

Fact 3: Two non-zero vectors are called parallel if and only if they point in exactly the same direction or they point in exactly opposite directions.

Fact 4: A non-zero vector is said to be parallel to a line if and only if the vector could be drawn entirely along the line.

Fact 5: A non-zero vector is said to be parallel to a plane if and only if the vector could be drawn entirely on the plane.

Fact 6: A non-zero vector is said to be perpendicular (or normal) to a plane if and only if the vector a right angle is formed every time it is drawn tail-to-tail with a non-zero vector parallel to the plane.

Vector Norms

Def 1: The norm of the vector \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) is \( |\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} \). In 2-dimensions and 3-dimensions this quantity is commonly referred to as the length of the vector. In applied problems this quantity is commonly referred to as the magnitude of the vector.

Fact 7: A vector, \( \vec{v} \), is called a unit vector if and only if \( |\vec{v}| = 1 \). Unit vectors are generally annotated thusly: \( \hat{v} \).

Fact 8: For a non-zero vector \( \vec{v} \), \( \hat{v} = \frac{\vec{v}}{|\vec{v}|} \) is the unit vector that points in the same direction as \( \vec{v} \).

The process of dividing a vector by its length is called normalizing the vector.

The Dot Product

Def 2: If \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \), then \( \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \).

Fact 9: The smallest angle, \( \theta \), formed by non-zero vectors \( \vec{u} \) and \( \vec{v} \) when \( \vec{u} \) and \( \vec{v} \) are drawn tail-to-tail satisfies the equation \( \cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \).
The Cross Product

Def 3: If \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \), then

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 \\
\end{vmatrix} = (u_2 v_3 - u_3 v_2) \hat{i} - (u_1 v_3 - u_3 v_1) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}.
\]

Fact 10: \( \vec{u} \times \vec{v} = - (\vec{v} \times \vec{u}) \)

Fact 11: For two non-zero, non-parallel vectors \( \vec{u} \) and \( \vec{v} \), \( \vec{u} \times \vec{v} \) is perpendicular to any plane that is parallel to both \( \vec{u} \) and \( \vec{v} \). That is, \( (\vec{u} \times \vec{v}) \perp \vec{u} \) and \( (\vec{u} \times \vec{v}) \perp \vec{v} \).

Vector Projections

Def 4 and 5: \( \text{comp}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{||\vec{u}||} \) and \( \text{proj}_{\vec{u}} \vec{v} = \left( \text{comp}_{\vec{u}} \vec{v} \right) \cdot \frac{\vec{u}}{||\vec{u}||} = \frac{\vec{v} \cdot \vec{u}}{||\vec{u}||^2} \cdot \vec{u} \)

Fact 12: \( \text{comp}_{\vec{u}} \vec{v} > 0 \iff \text{proj}_{\vec{u}} \vec{v} \uparrow \uparrow \vec{u} \) and \( \text{comp}_{\vec{u}} \vec{v} < 0 \iff \text{proj}_{\vec{u}} \vec{v} \uparrow \downarrow \vec{u} \)

Additional Fundamental Facts about Vectors, Lines, and Planes

Fact 13: Two non-zero vectors \( \vec{u} \) and \( \vec{v} \) form a right angle when drawn tail-to-tail if and only if \( \vec{u} \cdot \vec{v} = 0 \). Two such vectors are said to be perpendicular or orthogonal.

Fact 14: Two vectors \( \vec{u} \) and \( \vec{v} \) are parallel if and only if \( \vec{u} = k \vec{v} \) for some non-zero scalar \( k \).

Fact 15: A line that is parallel to the vector \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and passes through the point \( (x_0, y_0, z_0) \) can be modeled by the vector function \( \vec{r}(t) = \langle x_0 + u_1 t, y_0 + u_2 t, z_0 + u_3 t \rangle \) and the symmetric equations \( \frac{x - x_0}{u_1} = \frac{y - y_0}{u_2} = \frac{z - z_0}{u_3} \). The vector \( \vec{u} \) is called a direction vector for the line.

Fact 16: A plane that is perpendicular to the vector \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and passes through the point \( (x_0, y_0, z_0) \) can be modeled by the equation \( u_1 (x - x_0) + u_2 (y - y_0) + u_3 (z - z_0) = 0 \). The vector \( \vec{u} \) is called a normal vector for the plane.

Fact 17: The vector \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) is a normal vector for any plane with an equation of form \( u_1 x + u_2 y + u_3 z = k \).