1. Determine and state whether each given series is absolutely convergent, conditionally convergent, or divergent. Simply state your conclusion — no other work should be shown. (You may want to do some work on your scratch paper to help you decide your answer.) (21 points total)

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + 1} \text{ is } \underline{absolutely convergent} \]

\[ \sum_{k=1}^{\infty} \left( \frac{k}{k + 1} \right)^k \text{ is } \underline{divergent} \]

\[ \sum_{k=1}^{\infty} \frac{(-5)^{2k}}{k!} \text{ is } \underline{absolutely convergent} \]

\[ \sum_{k=1}^{\infty} \frac{(-1)^k + 1}{\ln(k + 1)} \text{ is } \underline{conditionally convergent} \]

\[ \sum_{k=1}^{\infty} \frac{\sin(\pi k)}{\sqrt[k]{k}} \text{ is } \underline{absolutely convergent} \]

\[ \sum_{k=1}^{\infty} \frac{(-5)^k}{(k + 1)6^k} \text{ is } \underline{absolutely convergent} \]

\[ \sum_{k=1}^{\infty} \frac{(-4)^k \sqrt{k + 1}}{k \cdot 4^k + 1} \text{ is } \underline{conditionally convergent} \]
2. Perform an absolute ratio test on the series \( \sum_{k=1}^{\infty} \frac{k^{2k}}{(2k)!} \) and state an appropriate conclusion. Make sure that all relevant work is presented in a manner consistent with that illustrated and discussed in class. (12 points)

Define \( a_k = \frac{k^{2k}}{(2k)!} \).

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(k+1)^{2(k+1)}}{(2k+2)!} \cdot \frac{(2k)!}{k^{2k}} \right|
\]

\[
= \lim_{k \to \infty} \left( \frac{(k+1)^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{k^{2k}} \right)
\]

\[
= \lim_{k \to \infty} \left( \frac{(k+1)^{2k} (k+1)^2}{2k+2} \cdot \frac{(2k)!}{(a+2)(2k+1)(2k)!} \right)
\]

\[
= \lim_{k \to \infty} \left( \frac{k+1}{2k+2} \cdot \frac{k+1}{2k+1} \cdot \left[ \left( \frac{k+1}{k} \right)^2 \right] \right)
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} \cdot e^2 \quad \text{Euler's Thm.}
\]

\[
= \frac{e^2}{4}
\]

Since \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| > 1 \), we know \( \sum_{k=1}^{\infty} a_k \) diverges.
3. Use an appropriate Taylor series template from Table 2 of “the lavender sheet” to help you determine a series whose value is equal to \( \int_0^{\frac{1}{2}} x^6 \sin(4x) \, dx \). To earn full credit, you must completely simplify the formula. Make sure that all relevant work is presented in a manner consistent with that illustrated and discussed in class. (14 points)

\[
\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \forall x.
\]

\[
\int_0^{\frac{1}{2}} x^6 \sin(4x) \, dx = \int_0^{\frac{1}{2}} x^6 \sum_{k=0}^{\infty} \frac{(-1)^k (4x)^{2k+1}}{(2k+1)!} \, dx
\]

\[
= \sum_{k=0}^{\infty} \int_0^{\frac{1}{2}} x^6 \frac{(-1)^k 4^{2k+1} x^{2k+1}}{(2k+1)!} \, dx
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1}}{(2k+1)!} \left[ \frac{x^{2k+2}}{2k+2} \right]_0^{\frac{1}{2}}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1}}{(2k+1)!} \left( \frac{1}{2} \right)^{2k+2}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} \left( \frac{1}{2} \right)^{2k+2}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k+1)!}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k-2}}{(2k+1)!}
\]
4. Determine the interval of convergence for the power series \( \sum_{k=1}^{\infty} \frac{(x-5)^k}{k \cdot 3^k} \). Make sure that all relevant work is presented in a manner consistent with that illustrated and discussed in class. 

(18 points)

Define \( a_k = \sum_{k=1}^{\infty} \frac{(x-5)^k}{k \cdot 3^k} \).

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{\frac{(x-5)^{k+1}}{(k+1) \cdot 3^{k+1}}}{\frac{(x-5)^k}{k \cdot 3^k}} \right| = \lim_{k \to \infty} \left| \frac{(x-5)^{k+1} \cdot k \cdot 3^k}{(k+1) \cdot 3^{k+1} \cdot (x-5)^k} \right| = \lim_{k \to \infty} \left| \frac{(x-5)^{k+1}}{(x-5)^k} \cdot \frac{k \cdot 3^k}{(k+1) \cdot 3^{k+1}} \cdot \frac{1}{3^k} \cdot \frac{1}{k+1} \right| = \lim_{k \to \infty} \left| \frac{x-5}{3} \cdot \frac{k}{k+1} \right| = \left| \frac{x-5}{3} \right|
\]

Since \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x-5}{3} \right| \), the AIT leads us to conclude that \( \sum_{k=1}^{\infty} a_k \) converges when \( \left| \frac{x-5}{3} \right| < 1 \) and diverges when \( \left| \frac{x-5}{3} \right| > 1 \). Sadly, the AIT is inconclusive when \( \left| \frac{x-5}{3} \right| = 1 \).

\[
\left| \frac{x-5}{3} \right| = 1 \Rightarrow x - 5 = 3 \text{ or } x - 5 = -3 \Rightarrow x = 8 \text{ or } x = 2.
\]

When \( x = 2 \), the series becomes \( \sum_{k=1}^{\infty} \frac{(1)^k}{k \cdot 3^k} \), which converges (AST).

When \( x = 8 \), the series becomes \( \sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k} \), which diverges (p-test).

The interval of convergence is \( I = [2, 8) \).
1. Use partial sum analysis to determine the value of \( \sum_{k=1}^{\infty} \frac{(-1)^k \cdot k^k}{(k^2)!} \) accurate through the seventh digit after the decimal point. Make sure that you present your work in a manner consistent with that exemplified during lecture. To earn full credit you must correctly identify the first two consecutive partial sums that round to the desired accuracy. (12 points)

You may **assume** that the series satisfies the conditions of the Alternating Series Remainder Theorem. It **can** be shown that the sequence generated by the absolute value of the terms is strictly decreasing and limits to 0 as \( k \to \infty \).

Since \( \sum_{k=1}^{\infty} \frac{(-1)^k \cdot k^k}{(k^2)!} \) satisfies the **ALT** (Alternating Series Test), all subsequent partial sum values fall between the values of any two consecutive partial sums. Through the seventh digit after the decimal point,

The actual series value falls

Through the seventh digit after the decimal point:

Since all values between the

Through actual value of \( S_2 + S_4 \) must round the same, through actual value of \( S_2 + S_4 \) must round the same, through

The seventh digit after the decimal point:

\[
\sum_{k=1}^{\infty} \frac{(-1)^k \cdot k^k}{(k^2)!} = -0.8334077
\]

2. Use the Taylor command on your calculator to determine the third degree Taylor Polynomial for the function \( f(x) = \sqrt{2x} \) centered at \( x = 2 \). Write the result below. No decimals! (3 points)

\[
T_3(x) = 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{64}
\]
3. For a certain function, \( f \), \( f(-5) = 3 \), \( f'(-5) = -1 \), \( f''(-5) = \frac{1}{2} \), and \( f'''(-5) = -48 \).

Write down the third degree Taylor polynomial for \( f \) centered at \( x = -5 \). Make sure that all of the coefficients are completely simplified. Do not expand your polynomial! (10 points)

\[
T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(-5)}{k!} (x + 5)^k
\]

\[
= \frac{2}{0!} + \frac{-1}{1!} (x + 5) + \frac{1/2}{2!} (x + 5)^2 + \frac{-48}{3!} (x + 5)^3
\]

\[
= 2 - (x + 5) + \frac{1}{4} (x + 5)^2 - 8 (x + 5)^3
\]

4. Khai takes one 50 mg tablet of Clearitout every morning, promptly at 10 am. Between doses, exactly 35% of the drug that was inside Khai immediately after he took the earlier pill is no longer there immediately before he takes the latter pill. Determine, over the long run, the exact amount of Clearitout that is inside Khai immediately after he takes a pill. Make sure that your reasoning is clear, that all relevant formulas and calculations are shown, and that your conclusion is clear. (10 points)

One approach...

The amount of drug inside Khai immediately after he takes the \( k \)th pill is modeled by the recursive sequence \( a_k = \begin{cases} 50 & \text{for } k = 1 \\ 50 + 0.65a_{k-1} & \text{for } k > 1 \end{cases} \)

Assuming that the sequence converges to \( L \), for large values of \( k \), \( a_k \approx a_{k-1} \approx L \). Thus:

\[
a_k = 50 + 0.65a_{k-1} \implies L = 50 - 0.65L
\]

\[
\implies 1.65L = 50
\]

\[
\implies L = 142.857 (\text{mg})
\]

Over the long run, immediately after Khai takes a pill, there are 142.857 mg of Clearitout in his body.