MTH 253 – Test 1  
Given: April 23, 2015  
Name: 

To earn full credit for any given problem, your work must be presented in a manner consistent with that illustrated and discussed in class; you must also get the correct answer to earn full credit for any given problem. *No notes or other references may be used. No calculator may be used.*

1. Find the sum of the telescoping series \( \sum_{k=1}^{\infty} \left( \frac{1}{2k+1} - \frac{1}{2k+3} \right) \). Make sure that you present your solution in a manner consistent with that illustrated in class. (12 points)

Define \( a_k = \frac{1}{2k+1} - \frac{1}{2k+3} \). Then:

\[
S_1 = a_1 = \frac{1}{3} - \frac{1}{5}
\]

\[
S_2 = S_1 + a_2 = \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{7} \right) = \frac{1}{3} - \frac{1}{7}
\]

\[
S_3 = S_2 + a_3 = \left( \frac{1}{3} - \frac{1}{7} \right) + \left( \frac{1}{7} - \frac{1}{9} \right) = \frac{1}{3} - \frac{1}{9}
\]

\[
\lim_{k \to \infty} S_k = \frac{1}{3} - \frac{1}{2k+3} \]

\[
\lim_{k \to \infty} S_k = \lim_{k \to \infty} \left( \frac{1}{3} - \frac{1}{2k+3} \right) = \frac{1}{3} - \frac{1}{3} = 0
\]

By definition, \( \sum_{k=1}^{\infty} \left( \frac{1}{2k+1} - \frac{1}{2k+3} \right) = \frac{1}{3} \)
2. Formally establish that the series \( \sum_{k=1}^{\infty} \frac{(-2)^{3k+4}}{5 \cdot 3^k \cdot 2^k} \) is geometric by simplifying the ratio of successive terms. Determine and state whether the series is convergent or divergent. If the series is convergent, do not find the sum. Make sure that you present your solution in a manner consistent with that illustrated in class. (12 points)

\[ a_k = \frac{(-2)^{3k+4}}{5 \cdot 3^k \cdot 2^k} \]

\[ \frac{a_{k+1}}{a_k} = \frac{(-2)^{3(k+1)+4}}{5 \cdot 3^{k+1} \cdot 2^{k+1}} \cdot \frac{5 \cdot 3^k \cdot 2^k}{(-2)^{3k+4}} \]

\[ = \frac{(-2)^{3k+7}}{3 \cdot 2^{k+1}} \cdot \frac{5 \cdot 3^k \cdot 2^k}{(-2)^{3k+4}} \]

\[ = \frac{(-2)^{3k+7}}{3 \cdot 2^{k+1}} \cdot \frac{5 \cdot 3^k \cdot 2^k}{(-2)^{3k+4}} \]

\[ = \frac{3^k \cdot 2^k}{3 \cdot 2^{k+1}} \cdot \frac{5}{(-2)^4} \]

\[ = -\frac{5}{8} \cdot \frac{3^k \cdot 2^k}{2^{k+1}} \]

Since \( a_k = \frac{-5}{8} \) and \( |r| < 1 \), the series converges.

3. Consider the separable differential equation \( (e^x + y) \frac{dy}{dx} = e^x - 2 \). Manipulate the equation into the form \( f(y) \, dy = g(x) \, dx \). Show all relevant work. That's it – do not go any further. Do not integrate. (8 points)

\[ (e^x + y) \frac{dy}{dx} = e^x - 2 \]

\[ e^x \frac{dy}{dx} + e^x \, dy = y \, dy - 2 \]

\[ e^x \frac{dy}{dx} + e^x \, dy = y \, dy - 2 \]

\[ \frac{e^x \, dy}{y \, dy - 2} = \frac{1}{e^x} \]

\[ \int \frac{e^{y+1}}{y - 2} \, dy = \int e^{-x} \, dx \]

Test 1
4. Find the value of \( \sum_{k=0}^{\infty} \left[ 6 \left( \frac{1}{3} \right)^k - 5 \left( \frac{1}{2} \right)^{k-1} \right] \). Make sure that you show all relevant calculations.

You do not need to be formal in your presentation, but you do need to include all relevant details that go into your determination. (12 points)

\[
\begin{align*}
\sum_{k=0}^{\infty} \left[ 6 \left( \frac{1}{3} \right)^k \right] & \quad a = a_0 = 6 \\
r & = \frac{1}{3} \\
\text{Sum: } \frac{a}{1-r} & = \frac{6}{1 - \frac{1}{3}} \\
& = 9
\end{align*}
\]

\[
\begin{align*}
\sum_{k=0}^{\infty} \left[ 5 \left( \frac{1}{2} \right)^{k-1} \right] & \quad a = a_0 = 10 \\
r & = \frac{1}{2} \\
\text{Sum: } \frac{a}{1-r} & = \frac{10}{1 - \frac{1}{2}} \\
& = 20
\end{align*}
\]

\[
\sum_{k=0}^{\infty} \left[ 6 \left( \frac{1}{3} \right)^k - 5 \left( \frac{1}{2} \right)^{k-1} \right] = 9 - 20 = -11
\]

5. Find the hundredth partial sum for the series \( \sum_{k=1}^{\infty} a_k \) where \( \{ a_k \} \) is the recursive sequence \( a_1 = 3, \ a_{k+1} = 2a_k - 3 \ \forall \ k \geq 1 \). Make sure that your reasoning is clear. (4 points)

\[
\begin{align*}
a_1 & = 3 \\
a_2 & = 2a_1 - 3 \\
& = 2(3) - 3 \\
& = 3 \\
a_3 & = 2a_2 - 3 \\
& = 2(3) - 3 \\
& = 3 \\
\end{align*}
\]

Clearly, \( a_k = 3 \ \forall k \).

\[
S_{100} = 100(3) = 300
\]
6. The sequence \( a_k = \begin{cases} 2 & \text{if } k = 1 \\ 4 - \frac{a_{k-1}}{7} & \text{if } k > 1 \end{cases} \) converges. Find the exact value to which the sequence converges. Show all relevant work and make sure that your thought process and conclusion are both clear. (8 points)

Let \( L \) be the number to which the series converges.

For large \( k \), \( a_k \approx a_{k-1} \approx L \).

\[
\begin{align*}
a_k &= 4 - \frac{a_{k-1}}{7} \\
\Rightarrow \quad L &= 4 - \frac{L}{7} \\
\Rightarrow \quad \frac{8}{7} L &= 4 \\
\Rightarrow \quad L &= \frac{7}{2}
\end{align*}
\]

\( \therefore \) The sequence converges to \( \frac{7}{2} \).

7. Sketch onto Figure 1 the solution curve that passes through the point \( (4, 0) \). (6 points)

![Figure 1: Slope field for the differential equation \( \frac{dy}{dx} = y \cos \left( \frac{x}{2} \right) + x \)](image)
8. For each series state whether the series is convergent or divergent and state a test you could perform to establish the convergence/divergence. Do not perform the test.

A correct answer for the first series has already been supplied to help you understand the directions. [20 points total]

<table>
<thead>
<tr>
<th>Series</th>
<th>Con/Di- verge?</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=1}^{\infty} \frac{3}{2 + 5k}$</td>
<td>Diverge</td>
<td>Limit Comparison Test with the p-series $\sum_{k=1}^{\infty} \frac{1}{k}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k} + 3^k}$</td>
<td>Converge</td>
<td>LCT with the geometric series $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k; \quad r = \frac{1}{3}; \quad</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} \frac{\sqrt{k^2 + k^4}}{k^2 + k^3}$</td>
<td>Diverge</td>
<td>LCT with the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{1/5}}; \quad p = \frac{2}{5}; \quad p &lt; 1$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} \frac{1}{k</td>
<td>\sin(k)</td>
<td>}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} \frac{k}{10^5}$</td>
<td>Converge</td>
<td>LCT with the geometric series $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k; \quad</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k} + \left(\frac{1}{3}\right)^k}$</td>
<td>Diverge</td>
<td>Divergence test. $\lim_{k \to \infty} \frac{\sqrt{k}}{\sqrt{k} + \left(\frac{1}{3}\right)^k} = 1 \neq 0$</td>
</tr>
</tbody>
</table>

\[ \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \text{ does not converge because} \]

\[ \lim_{k \to \infty} \frac{\sqrt{k}}{\sqrt{k} + \left(\frac{1}{3}\right)^k} = \lim_{k \to \infty} \frac{\left(3^k \sqrt{k}ight)}{\sqrt{k} + 3^k} = \frac{1/3}{1} = \frac{1}{3} \]

\[ = \infty \]
9. Circle T for true and F for false. (2 points each – 18 points total)

- T or F  \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is a \( p \)-series.

- T or F  \( \sum_{k=1}^{\infty} 5 \) is a geometric series.

- T or F  The divergence test is conclusive when applied to the series \( \sum_{k=1}^{\infty} \sin^2 (k) \).

- T or F  The divergence test is conclusive when applied to the series \( \sum_{k=1}^{\infty} \frac{1}{k} \).

- T or F  A direct comparison of the series \( \sum_{k=1}^{\infty} \frac{\sin^2 (k)}{k^4} \) to the convergent \( p \)-series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) establishes the convergence of \( \sum_{k=1}^{\infty} \frac{\sin^2 (k)}{k^4} \).

- T or F  A direct comparison of the series \( \sum_{k=1}^{\infty} \frac{\sin^2 (k)}{k} \) to the divergent \( p \)-series \( \sum_{k=1}^{\infty} \frac{1}{k} \) establishes the divergence of \( \sum_{k=1}^{\infty} \frac{\sin^2 (k)}{k} \).

- T or F  \( \sum_{k=1}^{\infty} ((-1)^k + (-1)^{k+1}) = \sum_{k=1}^{\infty} (-1)^k + \sum_{k=1}^{\infty} (-1)^{k+1} \)

- T or F  \( \sum_{k=1}^{\infty} ((-1)^k + (-1)^{k+1}) = \sum_{k=1}^{\infty} (-1)^k + \sum_{k=1}^{\infty} (-1)^{k+1} \)

- T or F  \( \sum_{k=1}^{\infty} \frac{3}{4^k} = 3 \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \)