1. List the first 4 terms in the sequence of partial sums for the series $\sum_{k=1}^{\infty} k!$; show the calculations. (7 points)

$$S_1 = a_1 = 1! = 1$$
$$S_2 = S_1 + a_2 = 1 + 2! = 1 + 2 = 3$$
$$S_3 = S_2 + a_3 = 3 + 3! = 3 + 6 = 9$$
$$S_4 = S_3 + a_4 = 9 + 4! = 9 + 24 = 33$$

2. List the first 4 terms in the sequence generated by $a_k = \begin{cases} 
2 & \text{if } k = 1 \\
3 & \text{if } k = 2 \\
6 - \frac{a_{k-1}}{3} - \frac{a_{k-2}}{2} & \text{if } k > 2
\end{cases}$

Show your calculations. (7 points)

$$a_1 = 2$$
$$a_2 = 3$$
$$a_3 = 6 - \frac{a_2}{3} - \frac{a_1}{2} = 6 - \frac{3}{3} - \frac{2}{2} = 4$$
$$a_4 = 6 - \frac{a_3}{3} - \frac{a_2}{2} = 6 - \frac{4}{3} - \frac{3}{2} = 1 \frac{1}{6}$$
3. Find the specific solution to the differential equation \( \frac{dy}{dx} = x^2 y + xy \) that passes through the point \((3,1)\). Make sure that you solve the solution equation for \( y \). (14 points)

\[
\begin{align*}
\frac{dy}{dx} &= x^2 y + xy \\
\frac{dy}{dx} &= y(x^2 + x) \\
\frac{dy}{y} &= (x^2 + x) \, dx \\
\int \frac{1}{y} \, dy &= \int (x^2 + x) \, dx \\
\ln(y) &= \frac{1}{3} x^3 + \frac{1}{2} x^2 + C \\
y &= e^{\frac{1}{3} x^3 + \frac{1}{2} x^2 + C} \\
y &= e^{\frac{1}{3} x^3 + \frac{1}{2} x^2} \cdot e^C \\
y &= C_1 e^{\frac{1}{3} x^3 + \frac{1}{2} x^2} \quad \text{is the general solution.} \\
y(3) = 1 &\implies 1 = C_1 e^{\frac{1}{3} \cdot 3^3 + \frac{1}{2} \cdot 3^2} \\
&\implies C_1 = e^{1 - \frac{27}{3} - \frac{9}{2}} \\
\text{The specific solution is:} \\
y &= e^{1 - \frac{27}{3} - \frac{9}{2}} e^{\frac{1}{3} x^3 + \frac{1}{2} x^2} \\
&\implies y = \frac{e^{\frac{1}{3} x^3 + \frac{1}{2} x^2}}{\sqrt{e^{\frac{1}{3} \cdot 3^3 + \frac{1}{2} \cdot 3^2}}} 
\end{align*}
\]
4. A slope field for each of the following differential equations is shown in one of figures 1 - 6.
For each equation, state the slope field (by figure number) that matches the equation.

\[ \frac{dy}{dx} = \ln(|y|) + \sin\left(\frac{\pi x}{3}\right) \] is shown in Figure 5.

\[ \frac{dy}{dx} = \ln(e^x) - \ln(|x|) \] is shown in Figure 4.

\[ \frac{dy}{dx} = \ln(|y|) - \cos\left(\frac{\pi y}{2}\right) \] is shown in Figure 3.
5. \( \sum_{k=0}^{\infty} \frac{4}{5^{k+1}} \) is a convergent geometric series; you do not need to prove this to me. What I need you to do is find the sum of the series; make sure that you show the calculation. (6 points)

\[
\begin{align*}
\frac{a}{1-r} &= \frac{a}{1-\frac{4}{5}} \\
&= \frac{a}{\frac{1}{5}} \\
&= 5a
\end{align*}
\]

6. \( \sum_{k=1}^{\infty} \frac{3 \cdot (-4)^{-2k+3}}{5^{-3k-1}} \) is a divergent geometric series. This you do need to prove to me. Make sure that you show me all of the necessary algebra and that you state the basis for your conclusion. (9 points)

Define \( a_k = \frac{2 \cdot (-4)^{-2k+3}}{5^{-3k-1}} \).

\[
\begin{align*}
\frac{a_{k+1}}{a_k} &= \frac{\frac{3 \cdot (-4)^{-2(k+1)+3}}{5^{-3(k+1)-1}}}{\frac{3 \cdot (-4)^{-2k+3}}{5^{-3k-1}}} \\
&= \frac{3 \cdot (-4)^{-2k+1}}{5^{-3k-4}} \cdot \frac{5^{-3k-1}}{3 \cdot (-4)^{-2k+3}} \\
&= \frac{(-4)^{-2k+1}}{(-4)^{-2k+3}} \cdot \frac{5^{-3k-1}}{5^{-3k-4}} \\
&= \frac{5^2}{(-4)^2} \cdot a_k \\
&= \frac{125}{16} \cdot a_k
\end{align*}
\]

So the series is geometric and the common ratio between terms is \( r = \frac{125}{16} \). Since \( |r| > 1 \), the series diverges.
7. Consider the telescoping series \( \sum_{k=1}^{\infty} \left[ \frac{4}{k+4} - \frac{4}{k+6} \right] \). Establish a pattern for the sequence of partial sums and state the formula for this pattern. Then use that formula to determine and state the sum of the series. (14 points)

Define \( a_k = \frac{4}{k+4} - \frac{4}{k+6} \). Then:

\[ S_1 = a_1 = \frac{4}{5} - \frac{4}{7} \]

\[ S_2 = S_1 + a_2 = \left( \frac{4}{5} - \frac{4}{7} \right) + \left( \frac{4}{6} - \frac{4}{8} \right) = \frac{4}{5} + \frac{4}{6} - \frac{4}{7} - \frac{4}{8} \]

\[ S_3 = S_2 + a_3 = \left( \frac{4}{5} + \frac{4}{6} - \frac{4}{7} - \frac{4}{8} \right) + \left( \frac{4}{7} - \frac{4}{9} \right) = \frac{4}{5} + \frac{4}{6} - \frac{4}{8} - \frac{4}{9} \]

\[ S_4 = S_3 + a_4 = \left( \frac{4}{5} + \frac{4}{6} - \frac{4}{8} - \frac{4}{9} \right) + \left( \frac{4}{8} - \frac{4}{10} \right) = \frac{4}{5} + \frac{4}{6} - \frac{4}{9} - \frac{4}{10} \]

\[ \vdots \]

\[ S_k = \frac{4}{5} + \frac{4}{6} - \frac{4}{k+5} - \frac{4}{k+6} \]

\[ \lim_{k \to \infty} S_k = \lim_{k \to \infty} \left( \frac{4}{5} + \frac{4}{6} - \frac{4}{k+5} - \frac{4}{k+6} \right) \]

\[ = \frac{4}{5} + \frac{4}{6} - 0 - 0 \]

\[ = \frac{22}{15} \]

\[ \therefore \text{By definition,} \quad \sum_{k=1}^{\infty} \left[ \frac{4}{k+4} - \frac{4}{k+6} \right] = \frac{22}{15} \]
8. Consider the series \( \sum_{k=1}^{\infty} \frac{2^k}{k^2 3^k} \). Perform a Limit Comparison Test with the series whose term formula is \( b_k = \frac{1}{k^2} \) and state an appropriate conclusion. Please note that an appropriate conclusion might be "the test was inconclusive." (8 points)

Define \( a_k = \frac{2^k}{k^2 3^k} \) and \( b_k = \frac{1}{k^2} \). Then:

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{2^k}{k^2 3^k}}{\frac{1}{k^2}} = \lim_{k \to \infty} \left( \frac{2^k}{3^k} \right) = 0
\]

Since \( \lim_{k \to \infty} \frac{a_k}{b_k} \) is finite and \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges (p-series, \( p = 2 > 1 \)), we can conclude based upon the LCT that \( \sum_{k=1}^{\infty} \frac{2^k}{k^2 3^k} \) converges.

9. \( \lim_{k \to \infty} \frac{\tan^{-1} \left( \frac{1}{k} \right)}{k} = \infty \). What, if anything, does this establish about the convergence or divergence of \( \sum_{k=1}^{\infty} \tan^{-1} \left( \frac{1}{k} \right) \)? Explain. (8 points)

\( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is convergent (p-series, \( p = 2 > 1 \)). If we define \( a_k = \tan^{-1} \left( \frac{1}{k} \right) \) and \( b_k = \frac{1}{k^2} \).

\( \lim_{k \to \infty} \frac{a_k}{b_k} = \infty \) means the LCT is inconclusive. This limit establishes nothing about the convergence or divergence of \( \sum_{k=1}^{\infty} \tan^{-1} \left( \frac{1}{k} \right) \).
10. For each series state whether the series is convergent or divergent and state a test you could perform to establish the convergence/divergence. Do not perform the test.

A correct answer for the first series has already been supplied to help you understand the directions. (3 points each)

<table>
<thead>
<tr>
<th>Series</th>
<th>Con/Diverge?</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{3}{2 + 5k} )</td>
<td>Diverge</td>
<td>Limit Comparison Test with the p-series ( \sum_{k=1}^{\infty} \frac{1}{k^p} )</td>
</tr>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k^2} + 3} )</td>
<td>Diverge</td>
<td>DCT or LCT with p-series ( \frac{1}{k} )</td>
</tr>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{1}{e^k} )</td>
<td>Converge</td>
<td>Geo Series with ( r = \frac{1}{e} ) 1 &lt; r &lt; 1</td>
</tr>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} )</td>
<td>Diverge</td>
<td>Divergence Test ( \lim_{k \to \infty} \frac{1}{\sqrt{k}} = 1 \neq 0 )</td>
</tr>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{5 + (-1)^k}{k^2} )</td>
<td>Converge</td>
<td>LCT with p-series where ( 1 &lt; p &lt; 2 )</td>
</tr>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{1}{(k!)^2} )</td>
<td>Converge</td>
<td>DCT or LCT with p-series ( \frac{1}{k^2} )</td>
</tr>
<tr>
<td>( \sum_{k=1}^{\infty} \frac{k \cdot 3^k}{5^k} )</td>
<td>Converge</td>
<td>Limit Comparison test with Geo Series ( \frac{3}{5} ) ( r^k ) where ( \frac{3}{5} &lt; r &lt; 1 )</td>
</tr>
</tbody>
</table>