Sum/Difference Property of Series and Multiplication Property of Series

If \( \sum_{k=N}^{\infty} a_k \) and \( \sum_{k=N}^{\infty} b_k \) both converge and \( c \) is any constant then:

\[ \sum_{k=N}^{\infty} (a_k \pm b_k) = \sum_{k=N}^{\infty} a_k \pm \sum_{k=N}^{\infty} b_k \quad \text{and} \quad \sum_{k=N}^{\infty} [c a_k] = c \sum_{k=N}^{\infty} a_k \]

If either \( \sum_{k=N}^{\infty} a_k \) or \( \sum_{k=N}^{\infty} b_k \) is divergent, the sum/difference property fails and if \( \sum_{k=N}^{\infty} a_k \) diverges the multiplication property fails.

For each of the following, determine if the equation is true or if it is false. If it’s false, state the reason the statement is false.

1. \( \sum_{k=0}^{\infty} [2 \cdot (-1)^k] = 2 \sum_{k=0}^{\infty} (-1)^k \)
   
   This is false; \( \sum_{k=0}^{\infty} (-1)^k \) does not converge.
   
   (Its sequence of partial sums is \( 1, 0, 1, 0, ... \))
   
   You can’t “do” \( \sum \) \( \# \) (a sum that doesn’t exist).

2. \( \sum_{k=1}^{\infty} \left[ \left( \frac{1}{2} \right)^k + k \right] = \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k + \sum_{k=1}^{\infty} k \)
   
   This is false; \( \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k \) does not exist.
   
   \( \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k = 1 \) but you still can’t “do” \( \sum \) \( \) \( \left( \frac{1}{2} \right)^k \) \( \) \( \) \( \) (does not exist)
Geometric Series can always be written in the form $a \cdot \left( \frac{1}{2} \right)^k$.

Find the value of $\sum_{k=0}^{\infty} \left( \frac{5}{2^k} - \frac{6 \cdot 2^k}{3^{k-1}} \right)$.

This is a geometric series with an $r$ value of $\frac{1}{2}$.

$a_{k+1} = \frac{1}{2} a_k \Rightarrow \frac{a_{k+1}}{a_k} = \frac{1}{2}$

$a = a_0 = 5$

$$\sum_{k=0}^{\infty} \frac{6 \cdot 2^k}{3^{k-1}} = \sum_{k=0}^{\infty} \frac{6 \cdot 2^k}{3^k \cdot 3^{-1}} = \sum_{k=0}^{\infty} \left[ 18 \cdot \left( \frac{2}{3} \right)^k \right] = \frac{18}{1 - \frac{2}{3}} = 54$$

$a = a_0 = 19$

$S$ since both series converge.

$$\sum_{k=0}^{\infty} \left[ \frac{5}{2^k} - \frac{6 \cdot 2^k}{3^{k-1}} \right] = \sum_{k=0}^{\infty} \frac{5}{2^k} - \sum_{k=0}^{\infty} \frac{6 \cdot 2^k}{3^{k-1}}$$

$$= 10 - 54$$

$$= -44$$

Find $r$

$$\sum_{k=0}^{\infty} \frac{3^{2k}}{5^{k-1}}$$

$r = \frac{9}{5}$

$a_k = \frac{3^{1k}}{5^{k-1}} = \frac{9^k}{5^{k-1}} = 5 \left( \frac{3}{5} \right)^k$

$$3^{2k} = (3^2)^k = \left( \frac{9}{5} \right)^k$$

$$b_k = \frac{3^{2k}}{5^{1-k}} = \frac{3^{2k}}{5^{1-k}} = \frac{2 \cdot 2^k}{5^{1-k}} = \frac{2 \cdot 2^k \cdot 5^k}{5^{2k}} = \frac{2 \cdot 2^k \cdot 5^k}{5^{2k}}$$

$$= \frac{2 \cdot 2^k \cdot 5^k}{5^{2k}}$$

$$= \frac{2 \cdot 2^k \cdot 5^k}{5^{2k}}$$

$$= \frac{2 \cdot (50)^k}{5^{2k}}$$
Introductory Example – The Integral Test

Use an appropriate improper integral to establish that \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \) diverges.

\[
\begin{align*}
\text{Areas} \quad & 
\beta_1 = \left( \frac{1}{\sqrt{1}} \right)(1) = \frac{1}{1} \\
& 
\beta_2 = \left( \frac{1}{\sqrt{2}} \right)(1) = \frac{1}{\sqrt{2}} \\
& 
\beta_3 = \left( \frac{1}{\sqrt{3}} \right)(1) = \frac{1}{\sqrt{3}} \\
\sum_{k=1}^{n} \frac{1}{\sqrt{k}} &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \\
& = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n \\
& = \sum_{k=1}^{n} \beta_k \\
\sum_{k=1}^{n} \beta_k &> \int_{1}^{n+1} \frac{1}{\sqrt{x}} \, dx \\
& \lim_{n \to \infty} \int_{1}^{n+1} x^{-1/2} \, dx = \lim_{n \to \infty} \frac{x^{1/2}}{1/2} \bigg|_{1}^{n+1} \\
& = \lim_{n \to \infty} \left[ 2\sqrt{n+1} - 2\sqrt{1} \right] \\
& = \infty \\
\therefore \quad \sum_{k=1}^{\infty} \beta_k &= \infty \\
\therefore \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \text{ diverges}
\end{align*}
\]
Introductory Example – The Integral Test

Use appropriate improper integrals to establish that \( \sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k^3}} \) converges.
Use the integral test to determine the convergence status of \( \sum_{k=2}^{\infty} \frac{\ln(k)}{k} \); remember that you have to establish that the test can be applied before you can apply the test.

By inspection, \( y = \frac{\ln(x)}{x} \) is continuous and positive

\[
\frac{dy}{dx} = \frac{1}{x} \cdot x - \frac{x \cdot \frac{d}{dx} \ln(x)}{x^2} = \frac{1 - \frac{\ln(x)}{x}}{x^2}
\]

\( \frac{dy}{dx} \leq 0 \) when \( 1 - \frac{\ln(x)}{x} < 0 \)

\[
\frac{dy}{dx} < 0 \quad \Rightarrow \quad \ln(x) > 1 \quad \Rightarrow \quad x > e
\]

\( \frac{\ln(x)}{x} \) is strictly decreasing, starting at \( x = e \) and \( \ln(x) \geq 2 \) for \( k \geq 3 \).

\( \therefore \) The integral test applies.

\[
\int_{2}^{\infty} \frac{\ln(x)}{x} \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{\ln(x)}{x} \, dx = \lim_{t \to \infty} \int_{2}^{t} \ln(x) \, du
\]

\[
= \lim_{t \to \infty} \left[ \frac{1}{2} [\ln(t)]^2 - \frac{1}{2} [\ln(2)]^2 \right] = 0
\]

\( \therefore \sum_{k=2}^{\infty} \frac{\ln(k)}{k} \) diverges.
$\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges: $p \leq 1$

Converges: $p > 1$

$k$ is the base (as opposed to $k$ being in the exponent).

State whether each series is divergent or convergent and state how you know.

$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ = $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

Diverges (p-series: $p = \frac{1}{2} \leq 1$)

$\sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$

Converges (P-series: $p = \frac{4}{3} > 1$)

$\sum_{k=1}^{\infty} \frac{1}{k^k}$

Converges (geometric: $r = \frac{1}{k} < 1$)

$\sum_{k=1}^{\infty} k^k$

Diverges

Divergence test ($\lim_{k \to \infty} a_k \neq 0$) ($\text{where } a_k = k^k$)
Prove the convergence or divergence of each of the following series.

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \]

The p-series diverges.

Go formal.

Define \( a_k = \frac{1}{k^2} \) and \( b_k = \frac{1}{k^2} \).

For \( k \geq 2 \), \( 0 < a_k \leq b_k \).

Since \( \sum_{k=1}^{\infty} b_k \) converges (p-series: \( p = 2 > 1 \)) by direct comparison, \( \sum_{k=1}^{\infty} a_k \) also converges.

\[ \frac{1}{6} - \frac{1}{12} + \frac{1}{24} - \frac{1}{48} + \ldots \]

The series is geometric with \( r = -\frac{1}{2} \).

Converges (1 < r < 1).

Integral Test/Inspection Tests/Comparison Tests — Section 8.3
Proof: By MVT, \( 0 \leq \frac{x}{k^2} \leq \frac{1}{k^2} \) for any \( x \) on \( [1/k, 1) \). By direct application of MVT, \( x/k \) converges (since \( x \to 1/k \)). Then, \( \sum k \geq 1 \) converges (by comparison test). Thus, \( \sum \frac{1}{k^2} \) is convergent (by the LCT). We know that \( \sum \frac{1}{k^2} \) is convergent (conjecture). Therefore, \( \frac{1}{k^2} \) is also convergent. This series is \( \sum \frac{2k+3}{k^2} \) or \( \sum \frac{2k+1}{k^2} \). Then \( \frac{2k+1}{k^2} \) is a conjecture. Mr. Simmons MTH 220

Thought Process: It is known that the series \( \sum \frac{1}{k^2} \) converges. So does \( \sum \frac{1}{k^2} \). Hence, \( \sum \frac{1}{k^2} \) is also convergent. This is because \( \sum \frac{1}{k^2} \) is convergent (conjecture). Therefore, \( \sum \frac{1}{k^2} \) is a conjecture.
Consider \( \sum_{k=1}^{\infty} \frac{k}{4^k} \).

Why can we not establish that the series converges using a direct comparison test with \( \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \)?

\[
\frac{k}{4^k} > \frac{1}{4^k} \quad \text{You cannot establish whether a series converges or diverges when you have numbers that are larger than those in a convergent sum.}
\]

Why can we not establish that the series converges using a limit comparison test with \( \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \)?

Define \( a_k = \frac{k}{4^k} \) and \( b_k = \frac{1}{4^k} \). Then

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{k}{4^k}}{\frac{1}{4^k}} = \lim_{k \to \infty} k = \infty
\]

All this tells us is that \( \frac{\frac{k}{4^k}}{\frac{1}{4^k}} \), the value of \( \frac{k}{4^k} \) are “looser” bigger than the value of \( \frac{1}{4^k} \).

How can we adjust “\( b_k \)” so that a limit comparison test is conclusive?

Let \( a_k = \frac{k}{4^k} \) and \( b_k = \frac{1}{2^k} \).

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{k}{4^k}}{\frac{1}{2^k}} = \lim_{k \to \infty} \left[ k \left( \frac{1}{2} \right)^k \right] = 0
\]
Prove that \( \lim_{x \to \infty} \left[ \left( \frac{3}{4} \right)^x \cdot x \right] = 0 \)

\[
\operatorname{lim}_{x \to \infty} \left[ \left( \frac{3}{4} \right)^x \cdot x \right] = \operatorname{lim}_{x \to \infty} \frac{x}{\left( \frac{3}{4} \right)^x} = \operatorname{lim}_{x \to \infty} \frac{\frac{dx}{dx} (x)}{\frac{dx}{dx} \left( \left( \frac{3}{4} \right)^x \right)} = \operatorname{lim}_{x \to \infty} \frac{1}{\ln \left( \frac{3}{4} \right) \cdot \left( \frac{3}{4} \right)^x} = 0
\]

Form: \( \frac{\infty}{\infty} \)

True or false – in each case explain. In all statements, assume that all terms of both series are positive.

a. If \( \sum_{k=1}^{\infty} b_k \) is known to converge and \( \lim_{k \to \infty} \frac{a_k}{b_k} = 0 \), we can **correctly** conclude that \( \sum_{k=1}^{\infty} a_k \) converges.

**False**: 

b. If \( \sum_{k=1}^{\infty} b_k \) is known to diverge and \( \lim_{k \to \infty} \frac{a_k}{b_k} = 0 \), we can **correctly** conclude that \( \sum_{k=1}^{\infty} a_k \) diverges.

**Correct**

c. If \( \sum_{k=1}^{\infty} b_k \) is known to diverge and \( \lim_{k \to \infty} \frac{a_k}{b_k} = 0 \), we can **correctly** conclude that \( \sum_{k=1}^{\infty} a_k \) converges.

**False**: 

d. If \( \sum_{k=1}^{\infty} b_k \) is known to converge and \( \lim_{k \to \infty} \frac{a_k}{b_k} = \infty \), we can **correctly** conclude that \( \sum_{k=1}^{\infty} a_k \) converges.

**False**: 

e. If \( \sum_{k=1}^{\infty} b_k \) is known to diverge and \( \lim_{k \to \infty} \frac{a_k}{b_k} = \infty \), we can **correctly** conclude that \( \sum_{k=1}^{\infty} a_k \) diverges.

**True**: 

f. If \( \sum_{k=1}^{\infty} b_k \) is known to converge and \( \lim_{k \to \infty} \frac{a_k}{b_k} = \infty \), we can **correctly** conclude that \( \sum_{k=1}^{\infty} a_k \) diverges.

**False**: 

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Integral Test/Inspection Tests/Comparison Tests – Section 8.3 | 9
4. For each of the following sequences, determine the value to which the sequence converges.

a. the sequence generated by \( a_k = \frac{\sqrt{4k^{16} + e^k}}{2^k + k^2} \)

\[
\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{\sqrt{4k^{16} + e^k}}{2^k + k^2} = \lim_{k \to \infty} \frac{\sqrt{e^k}}{2^k + k^2} = \lim_{k \to \infty} \frac{\sqrt{e^k}}{k^2 + 2^k} = 0
\]

The sequence converges to zero.

b. the sequence generated by \( b_k = \begin{cases} 
1 & \text{if } k = 1 \\
\frac{1 + b_{k-1}}{5 - b_{k-1}} & \text{if } k \geq 2
\end{cases} \)

Assume the sequence converges to \( L \), so that for large \( k \),

\[
b_{k-1} \approx b_k \approx L
\]

\[
b_k = \frac{1 + b_{k-1}}{5 - b_{k-1}} \implies L = \frac{1 + L}{5 - L}
\]

\[
5L - L^2 = 1 + L
\]

\[
L^2 - 4L + 1 = 0
\]

\[
L = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{2 + \sqrt{3}}{2}
\]

Convergent or divergent? How know? If a comparison test is needed, simply state the details of the test that needs to be applied (as well as the inevitable result, of course); do not go through the details of the test.

a. \( \sum_{k=1}^{\infty} \frac{5 \cdot 3^k}{2^{2k+1}} \)  

b. \( \sum_{k=1}^{\infty} \frac{5 + 3^k}{2^{2k+1}} \)  

c. \( \sum_{k=1}^{\infty} \frac{k^4}{k!} \)

d. \( \sum_{k=1}^{\infty} k^{-1.5} \)  

e. \( \sum_{k=1}^{\infty} \frac{1.5}{k} \)  

f. \( \sum_{k=1}^{\infty} \frac{1}{2\cos(k)} \)

g. \( \sum_{k=1}^{\infty} \frac{\cos(k)}{k^2} \)  

h. \( \sum_{k=1}^{\infty} \frac{\ln(4)}{e^k} \)  

i. \( \sum_{k=1}^{\infty} \frac{\ln(k)}{e^k} \)