The main topics considered in MTH 253 revolve around the two mathematical objects known as sequences and series. It is unfortunate that these words are so similar in appearance and sound because while the two objects are related they are completely different types of objects.

A sequence is an ordered set containing a never-ending list of numbers. A series is either a single number or is said to diverge.

A good way of thinking about most sequences is that they are never-ending lists of numbers generated by inserting the positive integers into some mathematical formula.

Find the first few terms of the sequence defined by \( a_k = 3 + \frac{(-1)^k}{k} \)

Find a formula for the sequence \(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{6}, \frac{7}{8}, -\frac{9}{10}, \ldots\)

Notice that the terms of the first sequence get closer and closer to 3 as you move through the list where-as the terms of the second sequence bounce back and forth between nearly 1 and nearly −1. This gives us our first peek at the difference between convergent sequences and divergent sequences.

**Definition**

If \( \lim_{n \to \infty} a_n = L \) we say that the sequence \( \{a_n\} \) converges to \( L \).

If \( \lim_{n \to \infty} a_n \) does not exist we say that the sequence \( \{a_n\} \) diverges; that is, a divergent sequence is a sequence that does not converge.
Let’s determine which of the following sequences are convergent and which are divergent. If the sequence converges, let’s state the number to which the sequence converges.

\[ a_k = \frac{3k + 1}{4k - 2} \]

Dominant Term Analysis

\[ b_k = \frac{3 \cdot 4^k - 39k^{50}}{20k^{50} + 4^k} \]

Dominant Term Analysis

\[ c_k = \frac{\sqrt{0.5^k + 5k^5 + 16k^{10}}}{\sqrt[3]{8k^{12} + 64}} \]

Dominant Term Analysis
**Definition**

A **recursive sequence** is a sequence where the value of the first term, and possibly a few terms following that term, are explicitly stated and the value of subsequent terms are defined by a formula based upon the values of the terms that precede them.

Let’s write out the first few terms of the sequence $a_k = \begin{cases} 3 & \text{if } k = 1 \\ 1 - \frac{a_{k-1}}{2} & \text{if } k > 1 \end{cases}$.

To what value does this sequence converge?

The Fibonacci sequence is the recursive sequence

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-2} + F_{k-1} & \text{if } k > 1 \end{cases}.$$

Let’s list the first few terms and illustrate a Fibonacci spiral with the drawing on the right.
Definitions

A **series** is the sum of the terms of a sequence.

The **$n$th partial sum** of a series is the sum of the first $n$ terms of the sequence being summed. That is, if the sequence being summed is \{ $a_k$ \}, the **$n$th partial sum** is $S_n = \sum_{k=1}^{n} a_k$ (assuming that $k$ starts at 1.)

Find the first four partial sums of the series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Notice that the sequence of partial sums is recursive in nature (if not in fact). For the series $\sum_{k=1}^{\infty} a_k$, we have:

\[
S_1 = a_1 \\
S_k = S_{k-1} + a_k \text{ if } k > 1
\]

Find the first five terms in the sequence of partial sums for the series $\sum_{k=1}^{\infty} \frac{1}{4^k}$. To what number does the sequence of partial sums appear to limit?
**Definitions**

A series is said to converge if its sequence of partial sums is convergent and the series is said to diverge if its sequence of partial sums diverges.

If the sequence of partial sums converges to the number $S$, we say that the sum of the series is $S$.

That is, we say that $\sum_{k=1}^{\infty} a_k = \lim_{k \to \infty} S_k = S$

**Telescoping Series** examples

Find the sum for the telescoping series $\sum_{k=1}^{\infty} \frac{4}{k^2 + 4k + 3}$. 

Use **proof by induction** to prove the partial sum formula from page 5.

1. Show that the formula is valid at $k = 1$

2. Assume that the formula is valid for $k$ and show that it follows from this that the formula is also valid for $k + 1$.

Establish the **partial fraction decomposition** on page 5 by hand.
Find the sum for the telescoping series \( \sum_{k=1}^{\infty} \frac{2+4k}{k^3+3k^2+2k} \).
Find the sum for the telescoping series \( \sum_{k=1}^{\infty} \ln \left( 1 + \frac{1}{k} \right) \).
Definition: *Geometric Sequence*

A sequence whose ratio of successive terms is constant is called a geometric sequence.

Determine which of the following formulas define geometric sequences.

\[ a_k = \frac{3 \cdot 2^{-k-1}}{5 \cdot (-3)^{-k+1}} \quad b_k = \frac{3 + 2^{k-1}}{5 \cdot (-4)^{k+1}} \quad c_k = \frac{3 \cdot 2^{k-1}}{5 \cdot (-4)^{k+1}} \]
What are the key features of the term formula for a geometric sequence?

Introductory example of a Geometric Series: Find the value of \( \sum_{k=1}^{\infty} \frac{3}{5^k - 1} \)
**Definition and Theorem: Geometric Series**

The sum of the terms of a geometric sequence is called a geometric series. Further more, if \( \frac{a_{k+1}}{a_k} = r \ \forall \ k \), then:

- The series diverges if \(|r| \geq 1\).
- The series converges if \(|r| < 1\). If the series converges then the sum of the series is \( \frac{a}{1 - r} \) where \( a \) is the first term of the series.

**Examples**

Show that the series is geometric and find the sum if it exists.

\[
\sum_{k=0}^{\infty} \frac{2^{-2k-1}}{5 \cdot (-3)^{-k+1}}
\]
\[ \sum_{k=1}^{\infty} \frac{2^{2k+11}}{4 \cdot (-2)^{5-3k}} \]

Find the value of \[ \sum_{k=0}^{\infty} \left( \frac{5}{2^k} - \frac{6 \cdot 2^k}{3^k-1} \right) \].