1. Determine which one of the slope fields on page 4 corresponds to the equation $\frac{dy}{dx} = -x y$. Then sketch onto the appropriate figure the solution curve that passes through the point $(0, -3)$. Finally, find the general solution to $\frac{dy}{dx} = -x y$ and determine that the equation for the specific solution you sketched is $y = \frac{-3}{\sqrt{e^{x^2}}}$. 

For spacing considerations, I’m going to find the symbolic solution first.

$$\frac{dy}{dx} = -x y \quad \Rightarrow \quad \frac{1}{y} \frac{dy}{dx} = -x$$

$$\Rightarrow \int \frac{1}{y} \, dy = \int -x \, dx$$

$$\Rightarrow \int \frac{1}{y} \, dy = \int -x \, dx$$

$$\Rightarrow \ln(|y|) = -\frac{x^2}{2} + C$$

$$\Rightarrow |y| = e^{-\frac{x^2}{2} + C} = e^{-\frac{x^2}{2}} \cdot e^C = C_1 e^{-\frac{x^2}{2}}$$

Let’s talk about the sign issue on $y$ for a bit. $|y|$ is always 1 of 2 things, $y$ or $-y$.

$y \geq 0 \Rightarrow |y| = y \Rightarrow y = C_1 e^{-\frac{x^2}{2}}$

$y \leq 0 \Rightarrow |y| = -y \Rightarrow -y = C_1 e^{-\frac{x^2}{2}} \Rightarrow y = C_2 e^{-\frac{x^2}{2}}$ where $C_2 = -C_1$

In either case, $y = (\text{some unknown constant}) e^{-\frac{x^2}{2}}$. For this reason, in the differential equations environment we tend to ignore the absolute value issue when computing $\int \frac{dy}{y}$ because we know that any negativity issue that emerges for a specific solution will work itself out when we solve for the unknown constant. So we’ll say that the general solution is $y = C_1 e^{-\frac{x^2}{2}}$.

Since $y = -3$ when $x = 0$ we have: $-3 = C_1 e^0 \Rightarrow C_1 = -3$.

So our specific solution is: $y = -3 e^{-\frac{x^2}{2}} \Rightarrow y = -3 \left(e^{x^2}\right)^{-1/2} \Rightarrow y = -\frac{3}{\sqrt{e^{x^2}}}$.
The slope field for \( \frac{dy}{dx} = -xy \) needs to have positive slopes in quadrants I and IV and negative slopes in quadrants I and III; this narrows the choices down to the figures 2, 4, and 6. At the point \((2,2)\), \( \frac{dy}{dx} = -xy = -4 \); clearly, Figure 2 is the only slope field that has this property. The solution curve is shown in Figure X.
2. Solve the initial value problem \( y'\sqrt{1+t^2} = -t - ty \); \( y(0) = 0 \) and show that the solution can be written in the form \( y = \frac{e}{e^{\sqrt{1+t^2}}} - 1 \).

\[
y'\sqrt{1+t^2} = -t - ty \quad \Rightarrow \quad \frac{dy}{dt} \sqrt{1+t^2} = -t(1+y)
\]

\[
\Rightarrow \quad \frac{1}{1+y} \frac{dy}{dt} = \frac{-t}{\sqrt{1+t^2}}
\]

\[
\Rightarrow \quad \int \frac{1}{1+y} \frac{dy}{dt} dt = \int \frac{-t}{\sqrt{1+t^2}} dt
\]

\[
\Rightarrow \quad \ln(1+y) = \int -w^{-1/2} \frac{dw}{2}
\]

\[
\Rightarrow \quad \ln(1+y) = -\frac{1}{2} w^{1/2} + C
\]

\[
\Rightarrow \quad \ln(1+y) = -\sqrt{1+t^2} + C
\]

\[
\Rightarrow \quad 1 + y = e^{-\sqrt{1+t^2} + c} = C_1 e^{-\sqrt{1+t^2}}
\]

\[
\Rightarrow \quad y = C_1 e^{-\sqrt{1+t^2}} - 1
\]

\[
\Rightarrow \quad 0 = C_1 e^{-1} - 1
\]

\[
\Rightarrow \quad C_1 = e
\]

So the specific solution is \( y = e \cdot e^{-\sqrt{1+t^2}} - 1 \Rightarrow y = \frac{e}{e^{\sqrt{1+t^2}}} - 1 \).
3. Solve the initial value problem \( y - y' \sec(x) = 0; \) \( y(\pi) = -e^2. \) Make sure that you state your solution in the form \( y = k e^{f(x)}. \)

\[
y - y' \sec(x) = 0 \quad \Rightarrow \quad y = \frac{dy}{dx} \sec(x)
\]

\[
\Rightarrow \quad \frac{1}{\sec(x)} = \frac{dy}{y} \frac{dx}{dx}
\]

\[
\Rightarrow \quad \int \cos(x) \, dx = \int \frac{dy}{y}
\]

\[
\Rightarrow \quad \int \cos(x) \, dx = \ln(y)
\]

\[
\Rightarrow \quad \sin(x) + C = \ln(y)
\]

\[
\Rightarrow \quad \ln(y) = \sin(x) + C
\]

\[
\Rightarrow \quad y = e^{\sin(x) + C} = C_1 e^{\sin(x)}
\]

\[
y(\pi) = -e^2 \quad \Rightarrow \quad -e^2 = C_1 e^{\sin(\pi)} = C_1 e^{0} = C_1
\]

So the specific solution is \( y = -e^2 e^{\sin(x)}. \)
4. E. Coli likes to divide; specifically, when placed into a welcoming host, each E. Coli cell divides into 2 cells every 20 minutes. Assuming that all cells divide on schedule and that no cells die, this means that when placed into a welcoming host an E. Coli population doubles every 20 minutes! If we let \( E \) be the number of E. Coli cells present in Gomer’s tummy \( t \) hours after he ingests 1500 E. Coli cells, and we assume that Gomer’s tummy is a welcoming host, then a formula for \( E \) can be found by solving the differential equation \( \frac{dE}{dt} = kE \). Find the formula for \( E \) and use it to determine how long it takes for Gomer's 1500 cell sample to grow into a 1,000,000 cell sample.

\[
\frac{dE}{dt} = kE \quad \Rightarrow \quad \frac{1}{E} \frac{dE}{dt} = k \quad \Rightarrow \quad \int \frac{1}{E} \frac{dE}{dt} dt = \int k dt \\
\Rightarrow \quad \int \frac{dE}{E} = \int k dt \\
\Rightarrow \quad \ln(E) = kt + C \\
\Rightarrow \quad E = e^{kt+C} = C_1 e^t
\]

\[y(0) = 1500 \quad \Rightarrow \quad 1500 = C_1 e^0 \]
\[\Rightarrow \quad C_1 = 1500\]

Alrighty, then… according to the constructs of the problem, after 20 minutes there are 3000 E. Coli cells in Gomer’s tummy; this means that \( y\left(\frac{1}{3}\right) = 3000 \). This gives us:

\[
3000 = 1500 e^{\frac{t}{3}} \quad \Rightarrow \quad t = 3 \ln(2) \\
\Rightarrow \quad y = 1500 e^{3\ln(2) t} \\
\Rightarrow \quad y = 1500 e^{(\ln(2)^3) t} \\
\Rightarrow \quad y = 1500 \cdot 2^{3t}
\]

Okie-dokie, we now need to determine when \( y = 1,000,000 \). Using our technology we get \( t \approx 3.12694059465 \). Further using our technology, we can conclude that it takes about 3 hours, 7 minutes, and 37 seconds for the population to grow to 1,000,000.
5. Yummy cookies were pulled from a 175°C oven and were left to cool in a 22°C room. In the first 5 minutes, one of the cookies cooled from 175°C to 99°C. Once removed from the oven, how much time (to the nearest second) did it take for this cookie to cool to within 5% of the room temperature?

\[ \frac{dT}{dt} = k(T - 22) \]

OK ... let's define \( T \) to be the temp of the of "the cookie" (°C) \( t \) seconds after it was removed from the oven. From Sir Newton's Law of Cooling we know that: \( \frac{dT}{dt} = k(T - 22) \). So...

\[
\int \frac{1}{T - 22} \, dt = \int k \, dt
\]

\[
\ln(T - 22) = kt + C
\]

\[
T - 22 = e^{kt + C} = Ce^{kt}
\]

\[
T = 22 + Ce^{kt}
\]

\[
y(0) = 175
\]

\[
y(300) = 99
\]

\[
k = \frac{\ln(77/153)}{300}
\]

Oh boy! \( T \) actually is greater than 22!

\[
y(0) = 175
\]

\[
y(300) = 99
\]

\[
T = 22 + 153e^{300k}
\]

\[
k = \frac{\ln(77/153)}{300}
\]

Okie dokie, our formula is \( T = e^{\frac{\ln(77/153)}{300}t} \). So let's determine when \( T = 1.05(22) = 23.1 \).

\[
23.1 = 22 + 153e^{\frac{\ln(77/153)}{300}t} \quad \Rightarrow \quad t \approx 2156.2
\]

Good golly, how many minutes are there in 2156 seconds??

It takes about 35 minutes, 56 seconds for that cookie to cool to within 5% of room temperature.
6. Find the specific solution for the curve in Figure 11.

\[
y^3 \frac{dy}{dx} + x^4 y \frac{dy}{dx} = x + x y^4
\]

\[
y^3 \frac{dy}{dx} + x^4 y \frac{dy}{dx} = x + x y^4 \Rightarrow \left(1 + x^4\right) y \frac{dy}{dx} = x (1 + y^4)
\]

\[
\Rightarrow \frac{y^3}{1 + y^4} \frac{dy}{dx} = \frac{x}{1 + x^4}
\]

\[
\Rightarrow \int \frac{y^3}{1 + y^4} \frac{dy}{dx} dx = \int \frac{x}{1 + x^4} dx
\]

\[
w = 1 + y^4
\]

\[
dw = 4 y^3 \ dy \Rightarrow \frac{dw}{4} = y^3 \ dy
\]

\[
\Rightarrow \int \frac{y^3}{1 + y^4} \frac{dy}{dx} dx = \int \frac{x}{1 + x^4} dx \Rightarrow \int \frac{dw}{4} = \int \frac{du/2}{1 + u^2}
\]

\[
\Rightarrow \frac{1}{4} \ln(w) = \frac{1}{2} \tan^{-1}(u) + C
\]

\[
\Rightarrow \ln(w) = 2 \tan^{-1}(u) + C_1
\]

\[
\Rightarrow w = e^{2 \tan^{-1}(u) + C_1} = C_2 e^{2 \tan^{-1}(x)}
\]

\[
\Rightarrow 1 + y^4 = C_2 e^{2 \tan^{-1}(x)}
\]

\[
y^4 = C_2 e^{2 \tan^{-1}(x)} - 1
\]

\[
y(0) = 0 \Rightarrow 0 = C_2 e^{2 \tan^{-1}(0)} - 1 = C_2 e^0 - 1
\]

\[
1 = C_2
\]

The specific solution is \( y^4 = e^{2 \tan^{-1}(x^2)} - 1 \).