This test is due at 5:30 pm on **Wednesday**, January 26, 2011.

You may not receive any help of any kind on any question on this test, nor may you receive any help of any kind on any question similar to a question on this test – get your practice homework done before you even look at the questions on this test. Please note that copying somebody else’s answer falls into the category of getting help with a question. Please note that failure to adhere to these rules will result in your teeth falling out while eating in an expensive restaurant, a roommate with a child who incessantly screams, and a sudden and persistent need for Kopectate.

You may ask Mr. Simonds if you do not understand what a question is asking. Please let him know as well if you think that the test has a substantive typo. I will post any necessary fixes on my web site.

To earn full credit on any given problem your work must be presented in a manner consistent with that demonstrated and discussed during both lecture and lab. Remember to show your algebraic steps one step at a time.

1. The three "sub-properties" that must be true about the function $f$ at the number $a$ if $f$ is continuous at $a$ are:

i. $f$ must be defined at $a$.  
ii. $\lim_{{x \to a}} f(x)$ must exist  
iii. $\lim_{{x \to a}} f(x)$ must equal $f(a)$

State each value of $x$ where the function $f$ shown in Figure 1 is discontinuous. For each discontinuity, state (by number) each of the sub-properties of continuity that fail at that value of $x$. (10 points)

<table>
<thead>
<tr>
<th>Location</th>
<th>Failing Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>i, ii, iii</td>
</tr>
<tr>
<td>-1</td>
<td>i, iii</td>
</tr>
<tr>
<td>4</td>
<td>i, ii, iii, iv</td>
</tr>
</tbody>
</table>

![Figure 1: $f$](image)
2. Use limit laws to formally establish the value of \( \lim_{x \to a} \frac{(2x-1)(x+4)}{(x-2)(3x-2)} \). (14 points)

To earn full credit your work must include all relevant limit law steps accompanied by a statement of the limit law number(s) used in each of those steps (as shown in class).

\[
\lim_{x \to a} \frac{(2x-1)(x+4)}{(x-2)(3x-2)} = \lim_{x \to a} \left( \frac{2x^2 + 7x - 4}{3x^2 - 8x + 4} \right) \frac{V_x L}{V_x L}
\]

\[
= \lim_{x \to a} \frac{2 + \frac{7}{x} - \frac{4}{x^2}}{3 - \frac{8}{x} + \frac{4}{x^2}}
\]

\[
= \frac{\lim_{x \to a} \left( 2 + \frac{7}{x} - \frac{4}{x^2} \right)}{\lim_{x \to a} \left( 3 - \frac{8}{x} + \frac{4}{x^2} \right)} \quad \text{LLAS}
\]

\[
= \frac{\lim_{x \to a} 2 + \lim_{x \to a} \frac{7}{x} - \lim_{x \to a} \frac{4}{x^2}}{\lim_{x \to a} 3 - \lim_{x \to a} \frac{8}{x} + \lim_{x \to a} \frac{4}{x^2}} \quad \text{LLAI, LLAS}
\]

\[
= \frac{2 + 0 - 0}{3 - 0 + 0} \quad \text{LLR}^1
\]

\[
= \frac{2}{3}
\]
3. Use limit laws to formally establish the value of \( \lim_{x \to \frac{3\pi}{4}} \frac{\cos(2x)}{\sin(x) + \cos(x)} \). (14 points)

To earn full credit, your work must include all relevant limit law steps accompanied by a statement of the limit law number(s) used in each of those steps (as shown in class).

\[
\lim_{x \to \frac{3\pi}{4}} \frac{\cos(2x)}{\sin(x) + \cos(x)} = \lim_{x \to \frac{3\pi}{4}} \frac{\cos^2(x) - \sin^2(x)}{\sin(x) + \cos(x)} \\
= \lim_{x \to \frac{3\pi}{4}} \frac{(\cos(x) + \sin(x))(\cos(x) - \sin(x))}{\sin(x) + \cos(x)} \\
= \lim_{x \to \frac{3\pi}{4}} (\cos(x) - \sin(x)) \\
= \cos(\frac{3\pi}{4}) - \sin(\frac{3\pi}{4}) \quad \text{LL A2} \\
= \cos(\frac{3\pi}{4}) - (-1) \quad \text{LL A6} \\
= \cos(\frac{3\pi}{4}) + \frac{\sqrt{2}}{2} \\
= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\
= -\sqrt{2}
\]
4. Consider the function \( g(x) = 6 + \pi \). (Yes, I really do mean \( \pi \).)
   a. What is the slope of any secant line connecting two points on \( g \)? (2 points)
      
      The slope is \( 0 \).
      
   b. Find and simplify the difference quotient for \( g \) showing each step in the simplification process. (8 points)
      
      The difference quotient for \( g(x) = 6 + \pi \) is:
      
      \[
      \frac{g(x+h) - g(x)}{h} = \frac{[6+\pi] - [6+\pi]}{h} = \frac{0}{h} = 0 \text{ for } h \neq 0
      \]
      
5. A function called \( z \) is shown in Figure 2. Suppose that you knew the formula for this function (you don't) and suppose that you used that formula to find the difference quotient of \( z \). What value would the difference quotient give you if you replaced \( x \) with \(-2\) and \( h \) with \( 4 \)? Show work that clearly indicates how you came up with your value. (6 points)
      
      If \( x = -2 \) and \( h = 4 \):
      
      \[
      \frac{z(x+h) - z(x)}{h} = \frac{z(-2) - z(-2)}{4} = \frac{2 - (-2)}{4} = 1
      \]
      
      Please note that 1 is the slope of the secant line drawn onto Figure 2.
6. Find and simplify the difference quotient for \( k(x) = x^3 - x \) showing each step in the simplification process. (12 points)

The difference quotient for \( k(x) = x^3 - x \) is:

\[
\frac{k(x+h) - k(x)}{h} = \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}
\]

\[
= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3 - h + x^3 - x}{h}
\]

\[
= \frac{h(3x^2 + 3xh + h^2 - 1)}{h}
\]

\[
= 3x^2 + 3xh + h^2 - 1 \text{ for } h \neq 0
\]
7. If a ball is thrown straight up into the air with an initial velocity of 40 ft/s, then the height (ft) of the ball \( t \) seconds after it is thrown is given by the function \( s(t) = 40t - 16t^2 \). What, including unit, is the value of \( s(2) \) and what does this value tell you in the context of the problem? (6 points)

\[ s(2) = 16 \text{ ft} \]

This tells us that two seconds after it was thrown the ball was 16 feet above the ground.

8. Let’s pretend that the speed of a rocket (measure in mph) \( t \) minutes after lift-off is given by the function \( s(t) = 120t \). The difference quotient for this function is constant. What, including unit, is this constant value and what does this value tell you in the context of the problem? (You do not need to show any work establishing the constant value of the difference quotient.) (6 points)

The difference quotient is constantly \( 120 \text{ mph/min} \).

This tells one that it was in the air, the speed of the rocket increased at the constant rate of \( 120 \text{ mph/min} \). That is, the speed increased by 120 mph with each passing minute.
9. Draw onto Figure 3 a function $f$, that satisfies each and every one of the following properties. Make sure that you draw and appropriately label all asymptotes. (14 points)

- The only discontinuities on $f$ occur at $-3$, $1$ and $3$.
- $f(-6) = 4$, $f(-3) = -1$, $f(0) = 2$, $f(1) = -1$
- $\lim_{x \to 3^-} f(x) = 2$, $\lim_{x \to 3^+} f(x) = \infty$, $\lim_{x \to \infty} f(x) = -3$
- $f$ is continuous from the right at $1$.
- $f$ has constant slope on $(-\infty, -3)$ and then again on $(-3, 1)$.

![Figure 3: f](image)

10. Consider the function $f(x) = \begin{cases} 
  x + 2 & \text{if } x < 0 \\
  \frac{2}{1 - 2x} & \text{if } 0 < x < 1 \\
  \frac{x - 2}{x^2 - 4} & \text{if } x \geq 1
\end{cases}$

State the values of $x$ where each of the following occur. If a stated property doesn't occur, make sure that you state that (as opposed to simply leaving the answer space blank). No explanation necessary. (2 points each)

a. At what values of $x$ is $f$ discontinuous?

$0$, $\frac{1}{2}$, $1$, and $2$

b. At what values of $x$ is $f$ continuous from the left but not the right?

There are none.

c. At what values of $x$ is $f$ continuous from the right but not the left?

At $1$.

d. At what values of $x$ does $f$ have removable discontinuities?

$0$ and $2$.