**The critical numbers** of \( f \) are the values of \( x \) in the domain of \( f \) where either \( f'(x) = 0 \) or \( f''(x) \) is undefined. These numbers are important because they are the only values of \( x \) where \( f \) could possibly have local extreme values.

**Example 1**
Consider the function \( f \) shown in Figure 1.

a. What are the critical numbers of \( f \)? Explain.

The domain of \( f \) is \((-\infty, 6) \cup (6, \infty)\).

\[ f'(x) = 0 \quad \text{at} \quad -7, 3, \text{and} \quad 9 \]

Over the domain of \( f \), \( f' \) is undefined at -6 and 6.

\( \therefore \) The critical numbers of \( f \) are \(-7, -6, 3, 9, \text{and} \quad 9 \).

b. Where does \( f \) change concavity? What else happens at these points?

\( f \) changes concavity at \(-6, 0, \text{and} \quad 8\). At these values of \( x \), \( f'' \) changes sign, so \( f' \) must either be zero or undefined.

c. What are the inflection points on \( f \)? Local max points? Local min points?

The inflection points are \((-6, 2)\) and \((0, 1)\) and \((9, 3)\). These occur at \(-6, 0, \text{and} \quad 9\).

The only local maximum point is \((-3, 4)\). This point occurs at \(-3\). The local maximum value is 4.

The local minimum points are \((-7, -2.5)\) and \((3, -2)\).
I see a Rational Exponent.

My Exponent tells me that the algebra will be Simpler if I use the product rule.

Example 2

Find the critical numbers of \( g(\theta) = \frac{\ln(\theta)}{\theta^{3/2}} \).

\[
 g'(\theta) = 2 \ln(\theta) \cdot \frac{1}{\theta} \cdot \theta^{-1/2} + \left( \frac{\ln(\theta)}{\theta^{1/2}} \right) \cdot \frac{1}{3} \theta^{-1/3}
\]

\[
 = 2 \ln(\theta) \cdot \theta^{-1/3} - \frac{1}{3} \left( \ln(\theta) \right)^2 \theta^{-1/3}
\]

\[
 = \frac{2 \ln(\theta)}{\theta^{1/3}} - \frac{\left( \ln(\theta) \right)^2}{3 \theta^{4/3}}
\]

\[
 = \frac{6 \ln(\theta) - \left( \ln(\theta) \right)^2}{3 \theta^{4/3}}
\]

The domain of \( g \) is \((0, \infty)\).

\( g'(\theta) = 0 \) where \( \ln(\theta) = 0 \) or \( 6 - \ln(\theta) = 0 \).

\[\begin{align*}
\ln(\theta) &= 0 \\
\theta &= e^0 \\
\end{align*}\]

\( g'(\theta) = 0 \) when \( \theta = 1 \) or \( \theta = e^6 \).

Over the domain of \( g \), \( g' \) is undefined nowhere.

\[\text{The critical nums of } g \text{ are } 1 \text{ and } e^6.\]
Example 3

Find the critical numbers of \( f(x) = (x - 4)^3 (x + 3)^3 \).

\[
f'(x) = 4(x-4)^3 \cdot 1 \cdot (x+3)^3 + (x-4)^3 \cdot 3(x+3)^2 \cdot 1 \\
= (x-4)^3 (x+3)^2 \left[ 4(x+3) + 3(x-4) \right] \\
= (x-4)^3 (x+3)^2 \left[ 7x \right] \\
= 7x (x-4)^3 (x+3)^2
\]

The domain of \( f \) is \(( -\infty, \infty )\).

\[ f'(x) = 0 \text{ at } 0, 4, \text{ and } -3. \]

\( f'(x) \) is never undefined.

\[ \therefore \text{ The critical numbers of } f \text{ are } 0, 4, \text{ and } -3. \]
Example 4

The first derivative of the function $T(x) = \frac{1100 \sqrt{x - 10}}{x^3}$ is $T'(x) = \frac{-5500(x - 12)}{3x^3 \sqrt{x - 10}}$.

a. What are the critical numbers of $T$? Explain.

The domain of $T$ is $(-\infty, 0) \cup (0, \infty)$.

$T'(x) = 0$ @ 12.

Over the domain of $T$, $T'$ is undefined @ 10.

The critical #s of $T$ are 10 and 12.

b. Build a table that clearly shows where $T$ is increasing and where $T$ is decreasing. Make sure your table communicates how you arrived at your conclusions.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$T''$</th>
<th>$x - 12$</th>
<th>$x^2$</th>
<th>$3\sqrt{x - 10}$</th>
<th>$T'$</th>
<th>$T$ is...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>-1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>decreasing</td>
</tr>
<tr>
<td>$[0, 10)$</td>
<td>7</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>$(10, 12)$</td>
<td>11</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>$(12, \infty)$</td>
<td>53</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

$c.$ What are the local maximum points on $T$? Local minimum points? What sort of point is $(10, 0)$?

The only local max - point is $(12, \frac{1100 \sqrt{2}}{144})$.

There are no local min points.

$(10, 0)$ is an inflection pt. that we stumbled upon. We’d need to analyze $T''$ to build an exhaustive list of inflection points.

4 Critical Numbers
Example 5

Find the local minimum points, local maximum points, and inflection points on the function 
\( h(x) = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}} \). Show all relevant work. There is additional work space on page 6.

\[
\begin{align*}
    h'(x) &= \frac{1}{3} x^{-\frac{2}{3}} - \frac{1}{3} x^{-\frac{4}{3}} \\
    h''(x) &= -\frac{2}{9} x^{-\frac{5}{3}} + \frac{4}{9} x^{-\frac{7}{3}} \\
    h'''(x) &= -\frac{1}{3} x^{-\frac{2}{3}} - \frac{1}{3} x^{-\frac{4}{3}} \\
    &= \frac{x^{\frac{2}{3}} - 1}{3 x^{\frac{4}{3}}} \\
    &= \frac{(x^{\frac{1}{3}} + 1)(x^{\frac{1}{3}} - 1)}{3 x^{\frac{4}{3}}} \\
    h''(x) &= -\frac{2}{9} \cdot \frac{1}{x^{\frac{5}{3}}} \cdot \frac{x^{\frac{4}{3}}}{x^{\frac{4}{3}}} + \frac{4}{9} \cdot \frac{1}{x^{\frac{7}{3}}} \\
    &= -\frac{2}{9} \cdot \frac{x^{\frac{1}{3}} - 2}{9 x^{\frac{7}{3}}} \\
    &= \frac{-2(x^{\frac{1}{3}} + \sqrt{2})(x^{\frac{1}{3}} - \sqrt{2})}{9 x^{\frac{7}{3}}} \\
\end{align*}
\]

The domain of \( h \) is \((-\infty, 0) \cup (0, \infty)\).

\( h'(x) = 0 \) when \( \sqrt[3]{x} = -1 \) or \( \sqrt[3]{x} = 1 \), so \( x = \pm 1 \).

Over the domain of \( h \), \( h' \) is never undefined.

\[ \because \text{The critical values of } h \text{ are } 1 \text{ and } -1. \]
\[ h(x) = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}} \quad h'(x) = \frac{(x^{\frac{1}{3}} - 1)(x^{\frac{1}{3}} + 1)}{3x^{\frac{4}{3}}} \quad h''(x) = \frac{2(\sqrt{2} + x^{\frac{1}{3}})(\sqrt{2} - x^{\frac{1}{3}})}{9x^{\frac{7}{3}}} \]

- \[ h''(x) = 0 \text{ when } 3x = \pm \sqrt{2}, \quad x = \pm \frac{2}{3} \]
- \[ h''(x) \text{ is undefined when } x = 0 \]

**Table 2: Increase Table for \( h \)**

<table>
<thead>
<tr>
<th>Interval</th>
<th>( \frac{1}{3} )</th>
<th>( 3x - 1 )</th>
<th>( 3x + 1 )</th>
<th>( x^{1/3} )</th>
<th>( h' )</th>
<th>Behavior of ( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty, -1)</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>(-1, 0)</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>(1, \infty)</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Note: End points in concavity test occur wherever \( h''(x) = 0 \) or \( h''(x) \) is undefined (over the range domain of \( h \)).

\[ h''(x) = 0 \text{ when } 3x = \pm \sqrt{2}, \quad x = \pm \frac{2}{3} \]

**Table 3: Concavity Table for \( h \)**

<table>
<thead>
<tr>
<th>Interval</th>
<th>( 3x + \frac{2}{3} )</th>
<th>( 3x - \frac{2}{3} )</th>
<th>( x^{1/3} )</th>
<th>( h'' )</th>
<th>Concavity of ( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -\frac{2}{3}))</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>up</td>
</tr>
<tr>
<td>((-\frac{2}{3}, 0))</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>down</td>
</tr>
<tr>
<td>((0, \frac{2}{3}))</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>up</td>
</tr>
<tr>
<td>((\frac{2}{3}, \infty))</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>down</td>
</tr>
</tbody>
</table>

The only local maximum point is \((-1, -2)\).

The only local minimum point is \((1, 2)\).

The inflection points are \((-2\sqrt{2}, -\frac{2\sqrt{2}}{2})\) and \((2\sqrt{2}, \frac{2\sqrt{2}}{2})\).
Example 6

Determine the intervals over which the function \( f'(\theta) = -\frac{\cos^2(\theta)}{2} - \cos(\theta) - \sin(\theta) - \theta \) is increasing and the intervals over which the function is decreasing.

The domain of \( f \) is \( \mathbb{R} \).

\[
 f'(\theta) = -\frac{1}{2} \cdot 2 \cos(\theta) \cdot (-\sin(\theta) + \sin(\theta)) - \cos(\theta) - 1 \\
 = \cos(\theta) \cdot \sin(\theta) + 3 \sin(\theta) - \cos(\theta) - 1 \\
 = \sin(\theta)(\cos(\theta) + 1) - 1(\cos(\theta) + 1) \\
 = (\cos(\theta) + 1)(\sin(\theta) - 1)
\]

\( \sin(\theta) \leq 1 \forall \theta \implies \sin(\theta) - 1 \) is never positive.

\( \cos(\theta) \geq -1 \forall \theta \implies \cos(\theta) + 1 \) is never negative.

\( \therefore \) The two factors in \( f' \) never have the same sign so their product is never positive. Ergo, \( f \) is never increasing, it's always decreasing (neither factor in \( f' \) is ever zero over an interval.)
Example 7

Find the critical numbers of \( k(t) = \frac{\sqrt{t-4}}{t-10} \).

\[ k'(t) = \frac{1}{2} (t-4)^{-\frac{1}{2}} (t-10)^{-1} + (t-4)^{\frac{1}{2}} (t-10)^{-\frac{3}{2}} - 1 \cdot (t-10)^{-2} \]

\[ = \frac{1}{2} \cdot \frac{1}{\sqrt{t-4}} \cdot \frac{1}{t-10} \cdot \frac{t-10}{2} - \frac{(t-4)^{\frac{1}{2}}}{(t-10)^{\frac{3}{2}}} \cdot \frac{2}{2} \cdot \frac{\sqrt{t-4}}{t-10} \]

\[ = \frac{(t-10) - 2(t-4)}{2 \sqrt{t-4} (t-10)^{\frac{3}{2}}} \]

\[ = \frac{-t - 2}{2 \sqrt{t-4} (t-10)^{\frac{3}{2}}} \]

\[ = -\frac{t + 2}{2 \sqrt{t-4} (t-10)^{\frac{3}{2}}} \]

The domain of \( k \) is \([-4, 10) \cup (10, \infty)\).

\( k'(t) = 0 \) nowhere (\(-2 \) is not in the domain of \( k' \) or \( k \)).

Over the domain of \( k \), \( k'(t) \) is undefined at \( 4 \).

i. The only critical # of \( k \) is \( 4 \).
For the function $g(x) = \frac{1-e^x}{1+e^x}$, $g'(x) = \frac{-2e^x}{(1+e^x)^2}$ and $g''(x) = \frac{2e^x(e^x-1)}{(1+e^x)^3}$. We are going to graph this function onto Figure 1 after answering each of the following questions.

- What are the critical numbers of $g$?
- What are the local minimum points and local maximum points on $g$?
- What are the inflection points on $g$?
- What are the asymptotes on a graph of $g$?

**The domain of $g$ is $(-\infty,\infty)$ ($e^x \neq -1$ over $\mathbb{R}$).**

- $g'(x)$ never equals zero and $g'$ is never undefined.
- $g$ has no critical numbers.
- If there aren't any critical numbers, there aren't any local extreme points. $g$ has no local extreme points.

- $g''(x) = 0$ when $e^x - 1 = 0$, which occurs at $0$.
- $g''(x)$ is never undefined.
  
  The only inflection point on $g$ is $(0,0)$.

- $g$ has no vertical asymptotes. ($e^x$ is always positive.)

- The horizontal asymptotes are $y = \lim_{x \to \infty} g(x)$ and $y = \lim_{x \to -\infty} g(x)$.

  - If these limits exist (numbers)

    \[ \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1-e^x}{1+e^x} = \lim_{x \to \infty} \left( 1 - \frac{1}{1+e^x} \right) = 1 \]

    \[ \lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{1}{1+e^x} = \frac{1}{1+1} = \frac{1}{2} \]

  - The horizontal asymptotes are $y = 1$ and $y = -1$.  

*Figure 1: $y = g(x)$*
MTH 251 – Mr. Simonds’ class

Find each limit showing work that fully supports your answer.

\[
\lim_{x \to \infty} \frac{(2x + 1)(3x - 2)}{5 - 7x^2} = \lim_{x \to \infty} \left( \frac{6x^2 - 6x - 2}{5 - 7x^2} \cdot \frac{1/x^2}{1/x^2} \right)
\]

\[
= \lim_{x \to \infty} \frac{6 - 1/x - 2/x^2}{5/x^2 - 7}
\]

\[
= \frac{6 - 0 - 0}{0}
\]

\[
\lim_{t \to 0} \frac{\sin^2(2t)}{1 - \cos^2(t)} = \lim_{t \to 0} \frac{4 \sin^2(t) \cos^2(t)}{1 - \cos^2(t)}
\]

\[
= \lim_{t \to 0} \frac{4 \sin^2(t)}{\sin^2(t)}
\]

\[
= 4
\]
\[
\lim_{h \to 0} \frac{h}{4 - \sqrt{16 + h}} = \lim_{h \to 0} \left( \frac{\frac{h}{4 - \sqrt{16 + h}}}{\frac{4 + \sqrt{16 + h}}{4 + \sqrt{16 + h}}} \right)
\]
\[
= \lim_{h \to 0} \frac{h \left( 4 + \sqrt{16 + h} \right)}{16 - (16 + h)}
\]
\[
= \lim_{h \to 0} \frac{h \left( 4 + \sqrt{16 + h} \right)}{-h}
\]
\[
= \lim_{h \to 0} \frac{4 + \sqrt{16 + h}}{-1}
\]
\[
= \frac{4 + \sqrt{16}}{-1}
\]
\[
= -8
\]

What are the asymptotes on a graph of the function \( y = \frac{4x}{\sqrt{x^3 - 6}} \)?

**Vertical asymptote:** \( x = \frac{3}{4} \) (not \(-2, 0\) or \(2\))

The only vertical asymptote is \( x = 6 \). \( x > 6 \) for both radicals to be real.

\( 3 \) is not in the domain of the function's radicals to be real.

**Horizontal asymptote:** \( \lim_{x \to \infty} f(x) = -\frac{4}{3} \)

**Domain:** the function is \( (6, \infty) \)

**Domain:** the function is \( (6, \infty) \)

If \( x > 6 \), then \( \sqrt{x} > \sqrt{6} \).

The horizontal asymptote is \( y = -\frac{4}{3} \).
Over what intervals is the function \( y = \frac{4x}{\sqrt{x-3} \sqrt{x-6}} \) increasing and over what intervals is the function decreasing?

\[
\frac{dy}{dx} = 4x \left( x-3 \right)^{-\frac{1}{2}} \left( x-6 \right)^{-\frac{1}{2}} + 4 \cdot \frac{1}{2} \left( x-3 \right)^{-\frac{3}{2}} \left( x-6 \right)^{-\frac{1}{2}} + 4 \cdot \left( x-3 \right)^{-\frac{1}{2}} \cdot \frac{1}{2} \left( x-6 \right)^{-\frac{3}{2}}
\]

\[
= \frac{4}{(x-3)^{\frac{1}{2}} (x-6)^{\frac{1}{2}}} - \frac{2x}{(x-3)^{\frac{3}{2}} (x-6)^{\frac{1}{2}}} - \frac{2x}{(x-3)^{\frac{1}{2}} (x-6)^{\frac{3}{2}}}
\]

\[
= \frac{4x^2 - 36x + 72}{(x-3)^{\frac{3}{2}} (x-6)^{\frac{1}{2}}} - \frac{2x^2 + 12x - 2x^2 + 6x}{(x-3)^{\frac{3}{2}} (x-6)^{\frac{1}{2}}}
\]

\[
= \frac{-18x + 72}{(x-3)^{\frac{3}{2}} (x-6)^{\frac{1}{2}}} - \frac{19 (x-4)}{(x-3)^{\frac{3}{2}} (x-6)^{\frac{3}{2}}}
\]

\[
\frac{dy}{dx} = 0 \quad \text{The function is always decreasing.}
\]