Introduction to Implicit Differentiation

Find the formula for the first derivative of the function $y = \sqrt{8-x^2}$ two ways.

1. Use Explicit Differentiation. (i.e. differentiate the explicitly stated function.)

\[
y = \sqrt{8-x^2} \implies y^2 = 8 \cdot x^2 \quad \text{and} \quad y \neq 0,
\]

\[
\frac{dy}{dx} = \frac{1}{2\sqrt{8-x^2}} \cdot \frac{d}{dx} (8-x^2)
\]

\[
= \frac{-1}{\sqrt{8-x^2}} \cdot 2x
\]

\[
= -\frac{x}{\sqrt{8-x^2}} = -\frac{x}{y}
\]

2. Use Implicit Differentiation. (i.e., differentiate the implicit equation $x^2 + y^2 = 8$.)

\[
x^2 + y^2 = 8
\]

\[
\frac{d}{dx} \left( x^2 + y^2 \right) = \frac{d}{dx} (8)
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
2y \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

A circle has different slopes at a single value of $x$, so the slope formula has to include both $x$ and $y$.

\[
\frac{dy}{dx} \bigg|_{(2,0)} = -\frac{2}{2} = -1
\]

\[
\frac{dy}{dx} \bigg|_{(2,1)} = -\frac{2}{-2} = 1
\]
MTH 251 – Mr. Simonds’ class

Introduction to the classic implicit differentiation strategy

Find the slope of the curve in Figure 2 at the point \((1,1)\). The equation used to graph the curve was

\[ y^5 + 3x^3y^2 + 5x^4 = 9 \]

1) Differentiate both sides of the equation with respect to \(x\).

- Remember to treat \(y\) as a function of \(x\) and use the chain rule. For example:
  \[ \frac{d}{dx}(y^2) = 2y \frac{dy}{dx} \]

2) Algebraically solve for \(\frac{dy}{dx}\).

3) Replace \(x\) and \(y\) with their values.

\[
\frac{d}{dx}(y^5) + \frac{d}{dx}(3x^3y^2) + \frac{d}{dx}(5x^4) = \frac{d}{dx}(9)
\]

\[
5y^4 \frac{dy}{dx} + 9x^2y^2 + 3x^3 \cdot 2y \frac{dy}{dx} + 20x^3 = 0
\]

\[
5y \frac{dy}{dx} + 9x^2 y^2 + 6x^3 \cdot \frac{dy}{dx} + 20x^3 = 0
\]

\[
(5y + 6x^2y) \frac{dy}{dx} = -9x^2y^2 - 20x^3
\]

\[
\frac{dy}{dx} = -\frac{9x^2y^2 + 20x^3}{5y + 6x^2y}
\]

The slope of the tangent line is:

\[
\left. \frac{dy}{dx} \right|_{(1,1)} = -\frac{24}{11} = -2\frac{4}{11}
\]

Therefore the equation of the tangent line at \((1,1)\) is:

\[
y = -\frac{24}{11}x + \frac{40}{11}
\]

2) Implicit Differentiation/Related Rates

\[
M = -3 \quad \text{pt.} \quad (2,7) \quad y = -3x + 13
\]
Find the equation of the tangent line to the curve in Figure 3 at the point \((-1, -1)\). The equation used to graph the curve was $2 \ln \left(\frac{x^2}{y^2}\right) + 2y = 2x + y \ln(x^2)$.

\[
\begin{align*}
2 \frac{\partial}{\partial x} \left(\frac{x^2}{y^2}\right) + 2y &= 2x + y \ln(x^2) \\
4 \frac{\partial}{\partial x} (x) - 4 \frac{\partial}{\partial y} (y) + 2y &= 2x + 2y \ln(x) \\
\frac{d}{dx} \left(4 x (x) - 4 y (y) + 2y\right) &= \frac{6}{dx} (2x + 2y \ln(x)) \\
\frac{4}{x} - \frac{4 dy}{dx} + 2 \frac{dy}{dx} &= 2 + 2 \frac{dy}{dx} \ln(x) + 2y \cdot \frac{1}{x} \\
(2 - \frac{y}{x} - 2 \ln(x)) \frac{dy}{dx} &= 2 + 2y \frac{1}{x} - \frac{y}{x} \\
\frac{dy}{dx} &= \frac{2 + 2y \frac{1}{x} - \frac{y}{x}}{2 - \frac{y}{x} - 2 \ln(x)} \\
\frac{dy}{dx} &= \frac{2 + 2y}{2x - y} \\
\text{The slope of the tangent line is given by} \\
\left.\frac{dy}{dx}\right|_{(-1,-1)} &= \frac{2 + 2y}{2 + y - 2 \ln(x)} \\
&= \frac{8}{6 - 0} \\
&= \frac{4}{3} \\
\therefore \text{The equation of the tangent line through} \\
\text{the point } (-1, -1) \text{ is} \\
y &= \frac{4}{3} x + \frac{1}{3}
\end{align*}
\]
Example 4

Let's use the fact that \( \frac{d}{dx} (\sec(x)) = \sec(x) \tan(x) \) to determine the derivative formula for the function \( y = \sec^{-1}(x) \). For simplicity's sake, let's assume that \( 0 < y < \frac{\pi}{2} \).

\[
\begin{align*}
    y &= \sec^{-1}(x) \\
    \Rightarrow \quad \sec(y) &= x \\
    \Rightarrow \quad \frac{d}{dx} (\sec(y)) &= \frac{d}{dx} (x) \\
    \Rightarrow \quad \sec(y) \tan(y) \frac{dy}{dx} &= 1 \\
    \Rightarrow \quad \frac{1}{\cos(y)} \cdot \frac{\sin(y)}{\cos(y)} \frac{dy}{dx} &= 1 \\
    \Rightarrow \quad \frac{dy}{dx} &= \frac{(\cos(y))^2}{\sin(y)}
\end{align*}
\]

Since \( \tan \) and \( \sec \) are reciprocal functions,

So if \( \sec(y) = x \), \( \cos(y) = \frac{1}{x} \)

\( \text{SOH} \quad \text{CAH} \quad \text{TIA} \)

\[
\begin{align*}
    \cos(y) &= \frac{1}{x} \quad (\text{CAH}) \\
    \sin(y) &= \frac{\sqrt{x^2-1}}{x} \quad (\text{SOH}) \\
    \Rightarrow \quad \frac{dy}{dx} &= \frac{(\cos(y))^2}{\sin(y)} \\
    \Rightarrow \quad \frac{dy}{dx} &= \frac{\left(\frac{1}{x}\right)^2}{\frac{\sqrt{x^2-1}}{x}} \\
    \Rightarrow \quad \frac{dy}{dx} &= \frac{1}{x \sqrt{x^2-1}} \quad (\text{VIOLATE})
\end{align*}
\]

\[
\begin{align*}
    \frac{d}{dx} (\sec^{-1}(x)) &= \frac{1}{x \sqrt{x^2-1}} \\
    \text{Note: } x \text{ is not always in Quadrant I.}
\end{align*}
\]
Example 5
Let’s use the process known as logarithmic differentiation to find a derivative formula for the function

\[ y = x^{\sin^2(x)} \]

\[
\frac{dy}{dx} = x^{\sin^2(x)} \cdot \ln(x) \\
\ln(y) = \ln(x^{\sin^2(x)}) \\
\frac{\ln(y)}{\ln(x)} = \sin^2(x) \cdot \ln(x) \\
\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left( \sin^2(x) \cdot \ln(x) \right) \\
\frac{1}{y} \frac{dy}{dx} = 2 \sin(x) \cdot \cos(x) \cdot \ln(x) + \frac{\sin^2(x)}{x} \\
\frac{dy}{dx} = y \left( 2 \sin(x) \cdot \cos(x) \cdot \ln(x) + \frac{\sin^2(x)}{x} \right) \\
= x^{\sin^2(x)} \left( 2 \sin(x) \cdot \cos(x) \cdot \ln(x) + \frac{\sin^2(x)}{x} \right)
Example 6

Let’s use logarithmic differentiation to find a derivative formula for the function $f(t) = \frac{\sqrt[5]{e^t \tan^{-1}(t)}}{6 \tan^3(t)}$.

\[
\ln(f(t)) = \ln\left( \frac{\sqrt[5]{e^t + \tan^{-1}(t)}}{6 \tan^3(t)} \right)
\]

\[
\ln(f(t)) = \ln\left( \left( e^t + \tan^{-1}(t) \right)^{1/5} \right) - \ln\left( 6 \tan^3(t) \right)^2
\]

\[
\ln(f(t)) = \frac{1}{5} \ln\left( e^t + \tan^{-1}(t) \right) - \ln(6) - 2 \ln(\tan(t))
\]

\[
\frac{d}{dt}(\ln(f(t))) = \frac{d}{dt}\left( \frac{1}{5} t + \frac{1}{5} \ln(\tan^{-1}(t)) - \ln(6) - 3 \ln(\tan(t)) \right)
\]

\[
\frac{1}{f(t)} \cdot \frac{d}{dt}(f(t)) = \frac{1}{5} + \frac{1}{5 \tan^{-1}(t)} \cdot \frac{d}{dt}(\tan^{-1}(t)) - 0 - 3 \cdot \frac{1}{\tan(t)} \cdot \frac{d}{dt}(\tan(t))
\]

\[
\frac{1}{f(t)} \cdot f'(t) = \frac{1}{5} + \frac{1}{5 \tan^{-1}(t)} \cdot \frac{1}{1+t^2} - 2 \cdot \frac{1}{\tan(t)} \cdot \sec^2(t)
\]

\[
f'(t) = f(t) \left( \frac{1}{5} + \frac{1}{5(1+t^2) \tan^{-1}(t)} - \frac{3 \sec^2(t)}{\tan(t)} \right)
\]

\[
= \frac{\sqrt[5]{e^t + \tan^{-1}(t)}}{6 \tan^3(t)} \left( \frac{1}{5} + \frac{1}{5(1+t^2) \tan^{-1}(t)} - \frac{3 \sec^2(t)}{\tan(t)} \right)
\]
Related Rates Example 1

A perfectly spherical snowball is melting at the constant rate of 3 cm³/min. At what rate are the radius, volume, and surface area changing at the instant the radius of the ball is 6 cm?

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>The volume of the snowball at time ( t )</td>
<td>cm³</td>
</tr>
<tr>
<td>( S )</td>
<td>The surface area of the snowball at that time</td>
<td>cm²</td>
</tr>
<tr>
<td>( r )</td>
<td>The radius of the snowball at that time</td>
<td>cm</td>
</tr>
<tr>
<td>( t )</td>
<td>The amount of time that has elapsed since melting began</td>
<td>min</td>
</tr>
<tr>
<td>( \frac{dv}{dt} )</td>
<td>The rate of change in ( V ) at ( t )</td>
<td>cm³/min</td>
</tr>
<tr>
<td>( \frac{dS}{dt} )</td>
<td>The rate of change in ( S ) at ( t )</td>
<td>cm²/min</td>
</tr>
<tr>
<td>( \frac{dr}{dt} )</td>
<td>The rate of change in ( r ) at ( t )</td>
<td>cm/min</td>
</tr>
</tbody>
</table>

**Diagram:**

**Equations that relate the non-rate variables:**

- \( V = \frac{4}{3} \pi r^3 \)
- \( S = 4 \pi r^2 \)
- \( \frac{dS}{dt} = 8 \pi r \frac{dr}{dt} \)

**Steps:**

1. **E:** \( V = \frac{4}{3} \pi r^3 \)
2. **D:** \( \frac{dV}{dt} = \frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right) \)
   \[ \frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} \]
3. **S:** Table 2: What we know when the radius is 6 cm

   | \( \frac{dV}{dt} \) | \( -3 \) (the volume is decreasing) |
   | \( r \) | 6 |
   | \( \frac{dr}{dt} \) | \( \text{unknown} \) |
   | \( \frac{dS}{dt} \) | \( \text{unknown} \) |

   \[ \left. \frac{dr}{dt} \right|_{r=6} = \frac{1}{4 \pi (6^3)^{\frac{2}{3}}} \cdot -3 \approx -0.007 \text{ cm/min} \]

Implicit Differentiation/Related Rate
The variable associated with a rate or a speed is always a derivative variable.

**Related Rates Example 2**

At noon one day a truck is 250 miles due east of a car. The truck is travelling west at a constant speed of 25 mph and the car is travelling due north at a constant speed of 50 mph. At what rate is the distance between the two vehicles changing 15 minutes after noon?

\[
\frac{dx}{dt} = -25, \quad \frac{dy}{dt} = 50
\]

**Table 3: Variable Definitions**

<table>
<thead>
<tr>
<th>Var.</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>The length of the indicated side opposite x</td>
<td>mi</td>
</tr>
<tr>
<td>y</td>
<td>The length of the indicated side opposite y</td>
<td>mi</td>
</tr>
<tr>
<td>z</td>
<td>The length of the indicated side opposite z</td>
<td>mi</td>
</tr>
<tr>
<td>t</td>
<td>The amount of time past noon</td>
<td>hr</td>
</tr>
<tr>
<td>(\frac{dx}{dt})</td>
<td>The rate that x changes over time</td>
<td>mi/hr</td>
</tr>
<tr>
<td>(\frac{dy}{dt})</td>
<td>The rate that y changes over time</td>
<td>mi/hr</td>
</tr>
<tr>
<td>(\frac{dz}{dt})</td>
<td>The rate that z changes over time</td>
<td>mi/hr</td>
</tr>
</tbody>
</table>

\[
x^2 + y^2 = z^2
\]

\[
\frac{d}{dt} (x^2 + y^2) = \frac{d}{dt} (z^2) \quad \implies \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}
\]

**Table 4: What we know at Quarter Past noon.**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Known / Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>250 - (\frac{1}{4}(25)) = 243.75</td>
</tr>
<tr>
<td>y</td>
<td>0 + (\frac{1}{4}(50)) = 12.5</td>
</tr>
<tr>
<td>z</td>
<td>(\sqrt{12.5^2 + 243.75^2}) = (\sqrt{59,570.7125})</td>
</tr>
<tr>
<td>(\frac{dx}{dt})</td>
<td>-25</td>
</tr>
<tr>
<td>(\frac{dy}{dt})</td>
<td>50</td>
</tr>
<tr>
<td>(\frac{dz}{dt})</td>
<td>Unknown</td>
</tr>
</tbody>
</table>

\[
\frac{dz}{dt} = \frac{\frac{dx}{dt} \frac{dy}{dt}}{z} = \frac{-25 \cdot 50}{\sqrt{59,570.7125}} = -22.4
\]

**Holy Moly!** The distance between the two cars is decreasing at a rate of 22.4 mph at 12:15 pm.
Related Rates Example 3

Sue Bob is one tall gal – 6 foot to be exact. Sue Bob is running towards a street light post; the lamp on the post is 40 ft above ground. The lamp is bright enough and the night is dark enough that Sue Bob casts a shadow behind her as she runs. When Sue Bob is exactly 30 feet from the base of the street light, she is running at exactly 10 ft/s. At what rate is the length of Sue Bob’s shadow changing at this instant? Is the shadow getting longer or shorter?

**Diagram:**

<table>
<thead>
<tr>
<th>Table 5: Variable definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Var.</strong></td>
</tr>
<tr>
<td>$x$</td>
</tr>
<tr>
<td>$y$</td>
</tr>
<tr>
<td>$b$</td>
</tr>
<tr>
<td>$\frac{dx}{dt}$</td>
</tr>
<tr>
<td>$\frac{dy}{dt}$</td>
</tr>
</tbody>
</table>

**Equation:**

From similar triangles: \[
\frac{6}{y} = \frac{40}{x+y}
\]

\[
6x = 3y \frac{dy}{dt}
\]

See where: \[
y = \frac{2}{17} x
\]

\[
\frac{dy}{dt} = \frac{2}{17} \frac{dx}{dt}
\]

Since \[
\frac{dy}{dt} \text{ always equals } \frac{2}{17} \frac{dx}{dt} \text{ and } \frac{dy}{dt} = -10 \text{ ft/s when }
\]

Sue Bob is 30 ft from street lamp and her shadow is decreasing at the rate of \[
\frac{20}{17} \text{ ft/s at that time.}
\]
Related Rates Example 4

A tank filled with water is in the shape of an inverted cone 20 feet high with a circular base (on top) whose radius is 5 feet. \( \frac{\text{Water is running out the bottom of the tank at the constant rate of } 2\ \text{ft}^3/\text{min}}{\text{How fast is the water level falling when the water is } 8\ \text{ft deep?}} \)

Table 7: Variable Table

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>Volume of remaining water, ( t ) minutes after drainage began</td>
<td>( \text{ft}^3 )</td>
</tr>
<tr>
<td>( h )</td>
<td>Height of water at ( t )</td>
<td>( \text{ft} )</td>
</tr>
<tr>
<td>( t )</td>
<td>Amount of time that has passed since</td>
<td>( \text{min} )</td>
</tr>
<tr>
<td>( \frac{dV}{dt} )</td>
<td>Rate at which ( V ) changes at that time</td>
<td>( \text{ft}^3/\text{min} )</td>
</tr>
<tr>
<td>( \frac{dh}{dt} )</td>
<td>Rate at which ( h ) changes at that time</td>
<td>( \text{ft/min} )</td>
</tr>
</tbody>
</table>

\[
V = \frac{\pi}{3} (\frac{1}{4})^2 h \Rightarrow V = \frac{\pi}{48} h^3
\]

\[
\frac{dV}{dt} = \frac{1}{16} h^2 \frac{dh}{dt}
\]

Table 8: When depth is \( 8 \) ft

<table>
<thead>
<tr>
<th>Variable</th>
<th>Known/unknown ( \frac{dh}{dt} = -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>8</td>
</tr>
<tr>
<td>( \frac{dh}{dt} ), ?</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{dh}{dt} = \frac{16}{\pi} \frac{dV}{dt} = -2, h = 8, \quad \Rightarrow \quad \frac{1}{2\pi}
\]

When the water depth is at 8 ft, it is falling at a rate of \( \frac{1}{2\pi} \) ft/min.