Notation, Notation, Notation

When working with function notation we use function notation to identify derivatives. This is the notation you’ve been seeing the past few weeks.

However, when working with equations that relate two variables, we use Leibniz Notation to identify derivatives.

Leibniz notation can get confusing because very similar symbols are used for things that are subtly, yet fundamentally different. The difference is akin to the difference between the verb, affect, and the noun, effect. The words are in many cases talking about one thing acting on another, but the word affect applies to the action where-as the word effect applies to the result of the action. For example, pressing down on a car’s accelerator affects the car’s speed and the effect is an increase in speed.

Similarly, taking the derivative affects the formula \( x^2 \) and the resultant effect is the formula \( 2x \). We use the symbols \( \frac{d}{dx}(x^2) \) to indicate that we are taking the derivative with respect to \( x \) of \( x^2 \). In a sense, the symbols \( \frac{d}{dx}(\ ) \) are telling us to do something—specifically, they tell us to take the derivative with respect to \( x \) of whatever lies within the parentheses.

On the other hand, if \( y = f(x) \), we say that \( \frac{dy}{dx} = f'(x) \). That is, \( \frac{dy}{dx} \) is the name we use to identify the result when we take the action of differentiating the formula for \( y \) with respect to \( x \).

Example 1

Write the Leibniz notation for each of the following and identify the result as a variable name or as a call to action.

“the derivative of \( z \) with respect to \( t \)”

\[
\frac{dz}{dt}
\]

“the derivative with respect to \( x \) of the square root of \( x \)”

\[
\frac{d}{dx}\left(\sqrt{x}\right)
\]
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\[ \frac{2}{y} = f'(x) \]
\[ \frac{\partial^2 z}{\partial y \partial x} \]

"the derivative with respect to \( t \) of \( \frac{dy}{dt} \)"

\[ \frac{d}{dt} \left( \frac{dy}{dx} \right) \]

"the second derivative of \( y \) with respect to \( t \)"

\[ f_\text{second derivative} \]
\[ \frac{d^2 y}{dx^2} = f'(x) \]

"the derivative with respect to \( x \) of the sine of \( y \)"

\[ \frac{d}{dx} \left( \sin(y) \right) \]

"the derivative with respect to \( t \) of \( V \) squared"

\[ \frac{d}{dt} \left( V^2 \right) \]

If \( y = x^2 \), then \( \frac{dy}{dx} = 2x \). We could establish this using the expression \( \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \), but if we went through that process (and didn’t mess it up) we’d end up with \( 2x \) so let’s not and just pretend that we did. In fact, let’s start skipping the limit process altogether and start riding the coattails of somebody else’s work. Other people have already gone through the limit process (and other processes) to establish the derivative formulas for all of the basic functions, so we might as well just memorize those formulas along with a few rules and get on with business.

The Power Rule: \[ \frac{d}{dx}(x^n) = nx^{n-1} \]

Example 2

Find \( \frac{d}{dx}(x^{12}) \) and \( \frac{d}{dt}(t^{-2}) \).

\[ \frac{d}{dx} (x^{12}) = 12x^{11} \]
\[ \frac{d}{dt} (t^{-2}) = -2t^{-3} \]
Rational Exponents

\[ \sqrt[n]{x^m} = x^{m/n} \]

The Constant Factor Rule: \( \frac{d}{dx}(k f(x)) = k \cdot \frac{d}{dx}(f(x)) \) for \( k \in \mathbb{R} \)

**Example 3**

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

\[ f(x) = \sqrt{x} \]

\[ f'(x) = \frac{d}{dx} \left( x^{1/2} \right) \]

\[ = \frac{1}{2} x^{-1/2} \]

\[ = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \]

\[ = \frac{1}{2\sqrt{x}} \]

\[ y = 4\sqrt{x^2} \text{ or } y = 4x^{2/3} \]

\[ \frac{dy}{dx} = 4 \cdot \frac{2}{3} x^{-1/3} \]

\[ = \frac{8}{3} \cdot \frac{1}{\sqrt[3]{x}} \]

\[ = \frac{8}{3\sqrt[3]{x}} \]

\[ z(t) = \frac{t^{10}}{3} \text{ (is a function of time)} \]

\[ z'(t) = \frac{1}{3} \cdot t^{9} \]

\[ z''(t) = \frac{9}{3} t^{8} \]

\[ \frac{dy}{dt} = z'(t) \]

\[ y = z(t) \]

\[ V = \frac{\pi r^2 h}{3} \text{; where } h \text{ is a constant} \]

\[ \frac{dV}{dr} = \frac{\pi h}{3} \cdot 2r \]

\[ = \frac{2\pi rh}{3} \]

\[ V' = \frac{2\pi rh}{3} \]

\[ V' = 2\pi rh \]

\[ \frac{d}{dt} \]

\[ \frac{dV}{dr} = \frac{\pi h}{3} \cdot 2r \]

\[ = \frac{2\pi rh}{3} \]

\[ 1CK!! \]
\( \omega (x) = (x + y)(x - y) \)
\( \omega (x) = x^2 - y \)

**The Sum/Difference Rule:**
\[
\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))
\]

\[
\frac{d}{dx}(k) = 0 \text{ for } k \in \mathbb{R}
\]
\[
\frac{d}{dx}(kx) = k \text{ for } k \in \mathbb{R}
\]

**Example 4**

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

\[ y = 4x + 7 \]
\[ \frac{dy}{dx} = 4 \]
\[ w(x) = 3x^2 - 8x + 4 \]
\[ w'(x) = 6x - 8 \]

\[ z = \frac{2x^2 - 5x + 4}{3} \]
\[ \frac{dz}{dx} = \frac{2}{3}x^2 - \frac{5}{3}x + \frac{4}{3} \]

\[ f(x) = 4 \sin(x) - \frac{2}{3} \cos(x) + 2 \arctan(x) \]
\[ f'(x) = 4 \cos(x) - \frac{1}{3} \cdot -\sin(x) + \frac{2}{1 + x^2} \]

\[ S = \pi r^2 h + 2\pi rh \text{ where } r \text{ is a constant} \]
\[ \frac{dS}{dh} = \pi r^2 + 2\pi r \]
The Product Rule: \[ \frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)) \]

Please note that on test 2 you will be expected to show me the Leibniz notation for the product rule and quotient rule each and every time you apply either rule.

Example 4

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

\[ z = \ln(t) \]
\[ \frac{dz}{dt} = \frac{d}{dt}(\ln(t)) + t \cdot \frac{d}{dt}(\ln(t)) \]
\[ = 1 \cdot \ln(t) + t \cdot \frac{1}{t} \]
\[ = \ln(t) + 1 \]

\[ y(x) = 3 \sin(x) \sin^{-1}(x) \]
\[ y'(x) = \frac{d}{dx}(3 \sin(x) \sin^{-1}(x)) = 3 \cos(x) \cdot \sin^{-1}(x) + \frac{3 \sin(x)}{\sqrt{1-x^2}} \]
\[ = 3 \cos(x) \sin^{-1}(x) + \frac{3 \sin(x)}{\sqrt{1-x^2}} \]

\[ p(\beta) = \frac{\sin(\beta) \cos(\beta)}{3} \]
\[ p(\beta) = \frac{1}{3} \sin(\beta) \cos(\beta) \]
\[ p'(\beta) = \frac{1}{3} \left[ \frac{d}{d\beta}(\sin(\beta) \cdot \cos(\beta)) + \sin(\beta) \cdot \frac{d}{d\beta}(\cos(\beta)) \right] \]
\[ = \frac{1}{3} \left[ \cos(\beta) \cos(\beta) + \sin(\beta) \cdot -\sin(\beta) \right] \]
\[ = \frac{1}{3} \left[ \cos^2(\beta) - \sin^2(\beta) \right] \]
\[ = \frac{1}{3} \cos(2\beta) \]
Example 5

Use the product rule to find \( \frac{d}{dx}(x^5 \cdot x^3) \) and show that the result makes sense.

\[
\frac{d}{dx}(x^5 \cdot x^3) = \frac{d}{dx}(x^5) \cdot x^3 + x^5 \cdot \frac{d}{dx}(x^3)
\]

\[
= 5x^4 \cdot x^3 + x^5 \cdot 3x^2
\]

\[
= 5x^7 + 3x^5
\]

\[
= 13x^5
\]

Example 6

Use the product rule to find \( \frac{d}{dt}(\sec(t) \cdot \cos(t)) \) and show that the result makes sense.

\[
\frac{d}{dt}(\sec(t) \cdot \cos(t)) = \frac{d}{dt}(\sec(t) \cdot \cos(t)) + \sec(t) \cdot \frac{d}{dt}(\cos(t))
\]

\[
= \sec(t) \cdot \tan(t) \cdot \cos(t) + \sec(t) \cdot (-\sin(t))
\]

\[
= \sec(t) \cdot \tan(t) \cdot \cos(t) - \sec(t) \cdot \sin(t)
\]

\[
= 0 \quad \text{Holy moly, the original function must be a constant function.}
\]

The Quotient Rule:

\[
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \cdot \frac{d}{dx}(g(x))}{[g(x)]^2}; \ g(x) \neq 0
\]

Example 7

Find \( \frac{d}{dx}\left(\frac{x^2}{\tan(x)}\right) \).

\[
\frac{d}{dx}\left(\frac{x^2}{\tan(x)}\right) = \frac{\frac{d}{dx}(x^2) \cdot \tan(x) - x^2 \cdot \frac{d}{dx}(\tan(x))}{[\tan(x)]^2}
\]

\[
= \frac{2x \cdot \tan(x) - x^2 \cdot \sec^2(x)}{\tan^2(x)}
\]
Example 8

Use the quotient rule to find \( \frac{d}{dt} \left( \frac{\sin(t)}{\cos(t)} \right) \) and show that the result makes sense.

\[
\frac{d}{dt} \left( \frac{\sin(t)}{\cos(t)} \right) = \frac{\frac{d}{dt}(\sin(t))\cos(t) - \sin(t)\frac{d}{dt}(\cos(t))}{(\cos(t))^2}
\]

\[
= \frac{\cos(t)\cos(t) - \sin(t)\sin(t)}{(\cos(t))^2}
\]

\[
= \frac{\cos^2(t) + \sin^2(t)}{(\cos(t))^2}
\]

\[
= \frac{1}{(\cos(t))^2}
\]

\[
= \frac{1}{\cos(t)}
\]

\[
= \sec(t)
\]

Example 9

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign.

Make sure that you use the appropriate name for each derivative.

\( f(x) = x^5 \sin(x) \cos(x) \)

\[
f'(x) = \frac{d}{dx} (x^5 \sin(x) \cos(x)) = x^5 \cos(x) \sin(x) + x^5 \sin(x) \cos(x) + x^5 \cos^2(x) - x^5 \sin^2(x)
\]

\[
= 6x^5 \sin(x) \cos(x)
\]

Please note that on test 2 you will be expected to show me the Leibniz notation for the product rule and quotient rule each and every time you apply either rule.
\[ y = \frac{e^t \ln(t)}{\sqrt{t^4}} \] (Use the “triple” product rule.)

\[
\frac{dy}{dt} = e^t \ln(t) \cdot \frac{1}{\sqrt{t^4}} + e^t \cdot \frac{\frac{d}{dt}(\ln(t))}{\sqrt{t^4}} + e^t \cdot \frac{\frac{d}{dt}(t^{1/2})}{\sqrt{t^4}}
\]

\[
= e^t \ln(t) \cdot \frac{1}{\sqrt{t^4}} + e^t \cdot \frac{\ln(t)}{t^{3/2}} + e^t \cdot \frac{1}{2t^{1/2}} \cdot \frac{1}{\sqrt{t^4}}
\]

\[
= e^t \ln(t) \cdot \frac{1}{\sqrt{t^4}} + e^t \cdot \frac{1}{t} \cdot \ln(t) \cdot \frac{7}{7t} - \frac{4e^t \ln(t)}{7} \cdot \frac{1}{\sqrt{t^4}}
\]

\[
= \frac{7 + e^t \ln(t) + 7e^t - 4e^t \ln(t)}{7t^{3/2}}
\]

\[ y = e^t \ln(t) \] (Start with the Quotient Rule and show that the result is the same as found the first time around.)

\[
\frac{dy}{dt} = \frac{\frac{d}{dt}(e^t \ln(t)) \cdot \frac{1}{\sqrt{t^4}} - (e^t \ln(t)) \cdot \frac{\frac{d}{dt}(\ln(t))}{\sqrt{t^4}}}{(\ln(t))^2}
\]

\[
= \frac{\frac{d}{dt}(e^t \ln(t)) + e^t \cdot \frac{\frac{d}{dt}(\ln(t))}{\sqrt{t^4}}}{\sqrt{t^4}} - e^t \ln(t) \cdot \frac{\frac{d}{dt}(\ln(t))}{\sqrt{t^4}}
\]

\[
= \frac{(e^t \ln(t) + e^t \cdot \frac{1}{t}) \cdot \ln(t) - e^t \ln(t) \cdot \frac{1}{\sqrt{t^4}}}{t^{3/2}}
\]

\[
= \frac{(e^t \ln(t) + e^t \cdot \frac{1}{t}) \cdot \ln(t) - e^t \ln(t) \cdot \frac{1}{\sqrt{t^4}}}{t^{3/2}}
\]

\[
= \frac{(7e^t \ln(t) + 7e^t - 4e^t \ln(t))}{7t^{3/2}}
\]