Definitions

*The difference quotient* for the function \( y = f(x) \) is the expression \( \frac{f(x + h) - f(x)}{h} \).

*The first derivative function* for \( f \) is \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).

Let’s discuss three fundamental differences between the difference quotient and the first derivative function.

1. The D.Q. finds an **average** rate of change.
   The first derivative finds the actual rate of change at a specific moment.

2. The difference quotient requires two values for evaluation \((x + h)\).
   The first derivative only requires one value for evaluation \(x\).

3. **Introduction to the first derivative**
   - The difference quotient finds the slope of a secant line.
   - The first derivative function finds the slope of a tangent line.
Alternate Definition

The first derivative function for \( y = f(x) \) is \( f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \).

Let’s use both of the definitions of the first derivative to find the first derivative formula for the function \( k(x) = x^3 \).

\[
k'(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}
\]

\[
= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}
\]

\[
= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}
\]

\[
= \lim_{h \to 0} (3x^2 + 3xh + h^2)
\]

\[
= 3x^2 + 3x(0) + 0^2
\]

\[
= 3x^2 \quad \text{Slope formula for } y = x^3
\]

For example, the tangent line
to \( y = x^3 \) when \( x = 2 \) has
a slope of \( k'(2) = 12 \).
The tangent line to the function $g$ in Figure 3 at the point where $t = -5$ has a slope of $\frac{1}{54}$. Use the definition of the first derivative at a point to verify this slope.

\[
\begin{align*}
g'(-5) &= \lim_{h \to 0} \frac{g(-5+h) - g(-5)}{h} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{4-(-5+h)}} - \frac{1}{\sqrt{4-(-5)}} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{9-h}} - \frac{1}{3} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{9-h}} \cdot \frac{2}{2} - \frac{1}{3} \cdot \frac{\sqrt{9-h}}{\sqrt{9-h}} \\
&= \lim_{h \to 0} \left( \frac{3 - \sqrt{9-h}}{3 \sqrt{9-h}} \cdot \frac{1}{h} \right) \\
&= \lim_{h \to 0} \left( \frac{3 - \sqrt{9-h}}{3 \sqrt{9-h}} \cdot \frac{3 + \sqrt{9-h}}{3 + \sqrt{9-h}} \right) \\
&= \lim_{h \to 0} \frac{9 - (9-h)}{3 \sqrt{9-h} (3 + \sqrt{9-h})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{9-h} (3 + \sqrt{9-h})} \\
&= \frac{1}{54}
\end{align*}
\]

Figure 3:
The function $g$ where $g(t) = \frac{1}{\sqrt{4-t}}$. 

Introduction to the first derivative | 3
Use the **first derivative function** to help you determine the vertex of the parabola \( y = g(x) \) where \( g(x) = 3x^2 - 5x + 23 \). Then use that formula to help you find the vertex of the parabola.

**Definition:**

\[
g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}
\]

\[
g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{(3(x+h)^2 - 5(x+h) + 23) - (3x^2 - 5x + 23)}{h}
\]

\[
= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 23 - 3x^2 + 5x - 23}{h}
\]

\[
= \lim_{h \to 0} \frac{6x + 3h - 5}{h}
\]

\[
= \lim_{h \to 0} \frac{h(6x + 3h - 5)}{h}
\]

\[
= \lim_{h \to 0} (6x + 3h - 5)
\]

\[
= 6x - 5
\]

At the vertex, the tangent line has a slope of zero.

\( 6x - 5 = 0 \)

\[ x = \frac{5}{6} \]

The vertex is \((\frac{5}{6}, \frac{25}{12})\)
The function $f$ is shown in figures A-D.

Draw onto Figure A the tangent lines to $f$ that have a slope of 0 and write into Table 1 the three implied values of $f'(x)$.

Figure B shows the tangent line to $f$ at $x = -1$. The slope of this line is $-1$. Write into Table 1 the two implied values of $f'(x)$.

Figure C shows the tangent line to $f$ at $x = 3$. The slope of this line is 1. Write into Table 1 the two implied values of $f'(x)$.

Figure D shows the tangent line to $f$ at $x = 2$. Write into Table 1 the two implied values of $f'(x)$.

Table 1: Tangent line slopes for $y = f(x)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>Figure #</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>use symmetry</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>use symmetry</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>use symmetry</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>

Sketch onto Figure 4 the function $f''$.

When $f'$ is:

$f$ is:
Let’s answer each of the following questions about \( g \) (Figure 5) over the interval \((-6, 6)\).

At what values of \( x \) does \( g'(x) = 0 \)?

At what values of \( x \) is \( g \) nondifferentiable?

Along what intervals is the value of \( g'(x) \) always positive?

Along what intervals is the value of \( g'(x) \) always negative?

Along what intervals is the value of \( g'(x) \) constant?

Three values of \( g'(x) \) are given in Table 2. Let’s go ahead and complete the rest of the table and then draw the function \( g' \) onto Figure 6.

Table 2: \( y = g'(x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2.01</th>
<th>-1.99</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y  )</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.6</td>
<td>1.5</td>
<td></td>
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</tbody>
</table>

Figure 6:
Computer drawn \( y = g(x) \)
Hand drawn \( y = g'(x) \)
Answer each question on this page in reference to the function \( y = f(x) \) shown in Figure 7. You may simply indicate the point(s) or intervals(s) in your response to questions 1-5.

1. At what value(s) of \( x \) does \( f''(x) = 0 \)?
   (The zeros occur at integers.)

2. Over what interval(s) is \( f''(x) \) positive?

3. Over what interval(s) is \( f'(x) \) negative?

4. Over what interval(s) is \( f' \) increasing? 5. Over what interval(s) is \( f'' \) decreasing?

6. Write a complete sentence that explains how you visually established the answer to question 1.

7. Write a complete sentence that explains how you visually established the answer to question 3.

8. Write a complete sentence that explains how you visually established the answer to question 4.
Figure 8 shows the tangent line to \( y = f(t) = \frac{2t}{t+1} \) at \((-3,3)\).

- Use the graph determine \( f'(3) \).
- Except at \(-1\), what must always be true about the function \( f' \)? How do you know?
- Where is \( f' \) increasing? Decreasing?
- Find the formula for \( f'(t) \).
- Verify your first two answers.

\[ t = -1 \]

![Graph showing tangent line to \( y = \frac{2t}{t+1} \) at \((-3,3)\).]