§ 4.1 Vector Spaces and Subspaces

Many of the algebraic properties of \( \mathbb{R}^n \) are shared by other mathematical systems that superficially seem quite different from \( \mathbb{R}^n \).

These algebraic properties are stated in the definition of a Vector Space.

**Definition:** A vector space is a non-empty set \( V \), whose elements are called vectors, together with two operations called vector addition and scalar multiplication where vector addition maps \( V \times V \rightarrow V \)

by \((\tilde{u}, \tilde{v}) \mapsto \tilde{u} + \tilde{v}\)

and scalar multiplication maps \( \mathbb{R} \times V \rightarrow V \)

by \((r, \tilde{u}) \mapsto r \cdot \tilde{u}\)

such that the following axioms are satisfied:

1. For all \( \tilde{u}, \tilde{v}, \tilde{w} \in V \) and all \( c, d \in \mathbb{R} \)
   1. \( \tilde{u} + \tilde{v} \in V \) (\( V \) is closed under vector addition)
   2. \( \tilde{u} + \tilde{v} = \tilde{v} + \tilde{u} \) (vector addition is commutative)
   3. \( (\tilde{u} + \tilde{v}) + \tilde{w} = \tilde{u} + (\tilde{v} + \tilde{w}) \) (vector addition is associative)
   4. There is a vector \( \tilde{0} \in V \) such that
      \[ \tilde{u} + \tilde{0} = \tilde{0} + \tilde{u} = \tilde{u} \] (additive identity element)
   5. For each \( \tilde{u} \in V \) there is a vector \( -\tilde{u} \in V \)
      such that \( \tilde{u} + (-\tilde{u}) = (-\tilde{u}) + \tilde{u} = \tilde{0} \) (additive inverses)
4.1.2

(6) \( c \cdot \vec{u} \in V \) (\( V \) is closed under scalar multiplication)

(7) \( c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v} \) (scalar multiplication distributes over vector addition)

(8) \((c+d)\vec{u}=c\vec{u}+d\vec{u} \) (scalar multiplication distributes over scalar addition)

(9) \( c(d\vec{u})=(cd)\vec{u} \) (scalar multiplication is associative)

(10) 1 \cdot \vec{u} = \vec{u} \) (identity property of scalar multiplication)

Using only these properties we can prove:

If \( V \) is a vector space:

(i) The zero vector, \( \vec{0} \), is unique.

(ii) For each \( \vec{u} \in V \), the additive inverse \( -\vec{u} \), or \( \vec{u} \) is unique.

Proof:

(i) Suppose there is another vector \( \vec{w} \in V \) such that \( \vec{u} + \vec{w} = \vec{w} + \vec{u} = \vec{0} \) for all \( \vec{u} \in V \).

Then \( \vec{w} = \vec{w} + \vec{0} \)

\[ = \vec{0}. \]

(ii) Let \( \vec{u} \in V \) be given. Suppose there is another vector \( \vec{w} \in V \) such that \( \vec{u} + \vec{w} = \vec{w} + \vec{u} = \vec{0} \).

Thus:

\[ \vec{w} = \vec{w} + \vec{0} \]

\[ = \vec{w} + (\vec{u} + (-\vec{u})) \]

\[ = (\vec{w} + \vec{u}) + (-\vec{u}) \]

\[ = \vec{0} + (-\vec{u}) \]

\[ = -\vec{u}. \]
Proposition: Let $V$ be a vector space and let $\vec{u} \in V$ and $c \in \mathbb{R}$, then

1. $0\vec{u} = \vec{0}$
2. $c\vec{0} = \vec{0}$
3. $-\vec{u} = (-1)\vec{u}$

Proof:

1. $0\vec{u} = (0+0)\vec{u}$
   
   $= 0\cdot\vec{u} + 0\cdot\vec{u}$  
   
   $0\vec{u} + (-0\vec{u}) = (0\cdot\vec{u} + 0\vec{u}) + (-0\vec{u})$  
   
   $\vec{0} = 0\cdot\vec{u} + (0\vec{u} + (-0\vec{u}))$  
   
   $\vec{0} = 0\cdot\vec{u} + \vec{0}$  
   
   $\vec{u} = 0\cdot\vec{u}$  

2. $c\vec{0} = c(\vec{0} + \vec{0})$
   
   $= c\vec{0} + c\vec{0}$  
   
   $c\vec{0} + (-c\vec{0}) = (c\vec{0} + c\vec{0}) + (-c\vec{0})$  
   
   $\vec{0} = c\vec{0} + (c\vec{0} + (-c\vec{0}))$  
   
   $\vec{0} = c\vec{0} + \vec{0}$  
   
   $\vec{0} = c\vec{0}$

3. $\vec{u} + (-1)\vec{u} = (1 + (-1))\vec{u}$

   $= 0\vec{u}$

   $= \vec{0}$ by (1).

   $\Rightarrow (-1)\vec{u} = -\vec{u}$ by the uniqueness of additive inverses.
Examples of vector spaces

1. $\mathbb{R}^n$ where $n \geq 1$, with the standard component-wise definition of vector addition and scalar multiplication is a vector space.
   (See §1.3, page 27, Algebraic Properties of $\mathbb{R}^n$)

2. Let $V$ be the set of all real valued functions on an interval $[a, b] \subseteq \mathbb{R}$, i.e., if $f \in V$, $f(x) \in \mathbb{R}$ for all $x \in [a, b]$.

   Define vector addition on $V$ by: If $f, g \in V$
   
   $f + g \in V$ is the function
   
   defined by
   
   $(f + g)(x) = f(x) + g(x) \in \mathbb{R}$
   
   for all $x \in [a, b]$.

   Define scalar multiplication on $V$ by: If $f \in V$ and $c \in \mathbb{R}$
   
   then $cf \in V$ is the function
   
   defined by
   
   $(cf)(x) = c \cdot (f(x)) \in \mathbb{R}$
   
   for all $x \in [a, b]$.

   The zero vector $\vec{0} \in V$ is the constant zero function
   
   $\vec{0}(x) = 0$ for all $x \in [a, b]$

   so if $f \in V$
   
   $(f + \vec{0})(x) = f(x) + \vec{0}(x)$
   
   $= f(x) + 0$
   
   $= f(x)$ for all $x \in [a, b]$

   $f + \vec{0} = f$ for all $f \in V$. 
2) If \( f \in V \) the additive inverse of \( f \) is the function
\[
-f = (-1)f \in V
\]
defined by \((-f)(x) = -(f(x)) \in \mathbb{R}^n\) for all \( x \in [a, b] \).

Then if \( f \in V \) and \( x \in [a, b] \)
\[
(f + (-f))(x) = f(x) + (-f)(x) = f(x) + (-f(x)) = 0 = \theta(x)
\]
so \( f + (-f) = \theta \).

The vector space axioms are all satisfied by the properties of real numbers:

Example: Axiom (1) \( c(f + g) = (cf) + (cg) \)
If \( f, g \in V \) and \( c \in \mathbb{R} \)
if \( x \in [a, b] \) then
\[
[c(f + g)](x) = c[f + g](x)
= c[f(x) + g(x)]
= c[f(x)] + c[g(x)]
= (cf)(x) + (cg)(x)
= [cf + cg](x)
\]
so \( c(f + g) = cf + cg \).
Example: For \( n \geq 0 \), let \( P_n \) be the set of all polynomials with real coefficients of the form

\[
p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n
\]

where the coefficients \( a_0, a_1, \ldots, a_n \) can be any real numbers and \( x \) is a real valued variable.

Note \( P_n \) is a subset of the set of all real valued functions on \((-\infty, \infty)\). We define vector addition and scalar multiplication on \( P_n \) the same way:

If \( p, q \in P_n \) and \( c \in \mathbb{R} \),

\[
p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n
\]

\[
q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n
\]

\[(p + q)(x) = p(x) + q(x) \quad \text{(collect like terms)} \]

\[(c \cdot p)(x) = c \cdot p(x) \quad \text{(scaling)} \]

and

\[(c \cdot p)(x) = c \cdot (p(x)) \]

\[= c(a_0) + (c_1a_1)x + (c_2a_2)x^2 + \cdots + (c_na_n)x^n\]

- The zero polynomial in \( P_n \) is

\[\overline{0}(x) = 0 \]

\[= 0 + (0)x + (0)x^2 + \cdots + (0)x^n\]

\[\min \{ (p + \overline{0})(x) \} = \]

\[= a_0 + (a_1 + 0)x + (a_2 + 0)x^2 + \cdots + (a_n + 0)x^n = \]

\[= p(x) \]

\[\Rightarrow p + \overline{0} = p \quad \text{for all} \quad p \in P_n \]
If $p \in \mathbb{R}^n$ the additive inverse of $p$ is $-p = (-1)p$

$$-p(x) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n$$

$$= (a_0 - a_0) + (a_1 - a_1)x + (a_2 - a_2)x^2 + \cdots + (a_n - a_n)x^n = 0 + 0x + 0x^2 + \cdots + 0x^n = \mathbb{O}(x)$$

$\Rightarrow p + (-p) = \mathbb{O}$

Again the vector space axioms are all satisfied by the properties of real numbers.

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(2) Let $W$ be the union of the first and third quadrants in the $xy$-plane, i.e. $W = \{[x, y] \in \mathbb{R}^2 \mid xy \geq 0\}$

If $\overrightarrow{u} \in W$ and $c$ is any scalar is $c\overrightarrow{u} \in W$? Why?

If $\overrightarrow{u} = [x, y] \in W$ then $xy \geq 0$

If $c \in \mathbb{R}$ then $c\overrightarrow{u} = [cx, cy]$ and $(cx)(cy) = c^2xy$ and $c^2 \geq 0$ and $xy \geq 0 \Rightarrow c^2(xy) \geq 0$ $

\Rightarrow c\overrightarrow{u} \in W$ for all $c \in \mathbb{R}$

$\Rightarrow W$ is closed under scalar multiplication.
4.1.8

(i) Find specific vectors $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v}$ is not in $W$.
This shows $W$ is not closed under vector addition (Axiom 1).
Hence $W$ is not a vector space.

Let $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$(3)(2) > 0 \Rightarrow \vec{u} \in W \\
(-2)(-3) > 0 \Rightarrow \vec{v} \in W

\vec{u} + \vec{v} = \begin{bmatrix} 3 - 2 \\ 2 - 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}

(1)(-1) = -1 < 0

$\Rightarrow \vec{u} + \vec{v} \notin W$

$\Rightarrow W$ is not a vector space.

Subspaces:

Let $V$ be a vector space.
A subspace of $V$ (sub-vector space) is a subset $H \subseteq V$ that satisfies the following three properties:

(a) $\vec{0} \in H$ (the zero vector of $V$ is contained in $H$)
(b) $H$ is closed under vector addition
   i.e. if $\vec{u}, \vec{v} \in H$ then $\vec{u} + \vec{v} \in H$ too.
(c) $H$ is closed under scalar multiplication.
   i.e. if $\vec{v} \in H$ and $c \in \mathbb{R}$ then $c\vec{v} \in H$ too.

Note: property (c) implies that if $\vec{v} \in H$, then $-\vec{v} = (-1)\vec{v} \in H$ too.

$\Rightarrow H$ is also closed under additive inverses.
It follows that all of the 10 vector space axioms are satisfied by \( H \) since they are satisfied by \( V \) and \( H \) is a subset of \( V \).

Therefore:
If \( H \) is a subspace of \( V \), \( H \) is a vector space in its own right under the operations of vector addition and scalar multiplication it inherits from \( V \).

Example: Let \( V \) be a vector space.
Let \( \mathbf{0} \in V \) be the zero vector.
Let \( H = \{ \mathbf{0} \} \) be the subset of \( V \) containing only the zero vector, then
\[
\begin{align*}
\text{(i) } & \mathbf{0} \in H \text{ is satisfied.} \\
\mathbf{0} + \mathbf{0} = \mathbf{0} \in H & \Rightarrow \text{(ii) is satisfied} \\
\text{If } c \in \mathbb{R} \text{ then } c \cdot \mathbf{0} = \mathbf{0} & \Rightarrow \\
\text{ } c \cdot \mathbf{0} \in H \text{ for all } c \in \mathbb{R} \Rightarrow \text{(iv) is satisfied} \\
\Rightarrow & H = \{ \mathbf{0} \} \text{ is a subspace of } V \\
\{ \mathbf{0} \} \text{ is called the trivial subspace of } V.
\end{align*}
\]
**Examples**

A plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

**Proof:**

A plane through the origin in $\mathbb{R}^3$ is a plane with equation of the form $ax+by+cz=0$.

Let $H = \{ [x, y, z] \in \mathbb{R}^3 | ax+by+cz=0 \}$.

Let $\mathbf{u} = \left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right] \in H$ because $a(x_1)+b(y_1)+c(z_1)=0$.

Let $\mathbf{v} = \left[ \begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array} \right]$.

Let both belong to $H$. Then

$a x_1 + b y_1 + c z_1 = 0$ and $a x_2 + b y_2 + c z_2 = 0$.

Claim: $\mathbf{u} + \mathbf{v} \in H$ too.

$\mathbf{u} + \mathbf{v} = \left[ \begin{array}{c} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{array} \right]$

and

$a(x_1+x_2) + b(y_1+y_2) + c(z_1+z_2) = 0$

$a x_1 + a x_2 + b y_1 + b y_2 + c z_1 + c z_2 = 0$

$(a x_1 + b y_1 + c z_1) + (a x_2 + b y_2 + c z_2) = 0 + 0 = 0$

$\mathbf{u} + \mathbf{v} \in H$ too.

Hence $H$ is closed under vector addition.

(2) Suppose $\mathbf{u} = \left[ \begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right] \in H$ and $c \in \mathbb{R}$

Then $a x_1 + b y_1 + c z_1 = 0$.

Now $c \mathbf{u} = \left[ \begin{array}{c} c x_1 \\ c y_1 \\ c z_1 \end{array} \right]$.
4.1.1

\[ a(cx_1) + b(ry_1) + c(z_1) = \]
\[ n(ax_1) + r(by_1) + l(z_1) = \]
\[ \sqrt{ax_1 + by_1 + cz_1} = \]
\[ \sqrt{0} = 0 \]
\[ \Rightarrow \sqrt{y} \in H \text{ too.} \]
\[ \Rightarrow H \text{ is closed under scalar multiplication} \]

\( \Rightarrow H \text{ is a subspace of } \mathbb{R}^3 \)

(5) Determine if the given set is a subspace of \( \mathbb{R}^n \) for an appropriate value of \( n \).

(6) All polynomials of the form \( p(t) = a + t^2 \) where \( a \in \mathbb{R} \), \( p(t) = a + (0)t + (1)t^2 \)
\[ \bar{0} = 0 + 0t + 0t^2 \]
is not of the form \( a + t^2 \) for some \( a \in \mathbb{R} \)
\[ \Rightarrow \bar{0} \notin \mathbb{R} = a + t^2 / a \in \mathbb{R}^3 \]
\[ \Rightarrow H = \mathbb{R} + t^2 / a \in \mathbb{R}^3 \]
is not a subspace
it fails to contain the zero vector.

(8) All polynomials such that \( p(0) = 0 \)
\[ H = \mathbb{R} p \in P_n / p(0) = 0 \]
(8) \( \bar{0}(0) = 0 \Rightarrow \bar{0} \in H \).
(9) If \( p, q \in H \) then \( p(0) = 0 \) and \( q(0) = 0 \)
\[ \Rightarrow (p + q)(0) = p(0) + q(0) = 0 \]
\[ \Rightarrow p + q \in H \]
\[ \Rightarrow H \text{ is closed under vector addition}. \]
4.1.12

(3) If \( p \in \mathbb{R}^1 \) and \( c \in \mathbb{R} \) then
\[
 p(0) = 0 \implies (cp)(0) = c (p(0)) = c (0) = 0.
\]
\[
\implies cp \in H
\]
\[
\Rightarrow H \text{ is closed under scalar multiplication.}
\]

(4) \( c(0) = H \) is a subspace of \( \mathbb{R}^n \).

Let \( V \) be a vector space.

Let \( S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) be a set of vectors in \( V \). Let \( c_1, c_2, \ldots, c_k \in \mathbb{R} \) be scalars. Then a linear combination of the vectors in \( S \) is a vector of the form:
\[
\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k.
\]

**Definition:** Let \( V \) be a vector space. Let \( S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) be a set of vectors in \( V \). Then the span of \( S \), denoted \( \text{Span}(S) \) or \( \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) \), is the set of all possible linear combinations of the vectors in \( S \).
\[
\text{Span}(S) = \{\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.
\]

**Theorem (1):** If \( V \) is a vector space and \( S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) is a non-empty subset of vectors in \( V \), then
\[
H = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) \text{ is a subspace of } V.
\]
Proof:

(a) \( \tilde{0} = 0 \tilde{v}_1 + 0 \tilde{v}_2 + \ldots + 0 \tilde{v}_k \in H \).

(b) If \( \tilde{u} = c_1 \tilde{v}_1 + c_2 \tilde{v}_2 + \ldots + c_k \tilde{v}_k \in H \)
and \( \tilde{v} = d_1 \tilde{v}_1 + d_2 \tilde{v}_2 + \ldots + d_k \tilde{v}_k \in H \)
then \( \tilde{u} + \tilde{v} = (c_1 d_1) \tilde{v}_1 + (c_2 + d_2) \tilde{v}_2 + \ldots + (c_k + d_k) \tilde{v}_k \in H \).

(\Rightarrow) If \( \tilde{u}, \tilde{v} \in H \) then \( \tilde{u} + \tilde{v} \in H \) too.

(\Rightarrow) \( H \) is closed under vector addition.

(c) If \( \tilde{u} = c_1 \tilde{v}_1 + c_2 \tilde{v}_2 + \ldots + c_k \tilde{v}_k \) and \( r \in \mathbb{R} \)
then \( r\tilde{u} = r(c_1 \tilde{v}_1 + c_2 \tilde{v}_2 + \ldots + c_k \tilde{v}_k) \)
\[ = r(c_1 \tilde{v}_1) + r(c_2 \tilde{v}_2) + \ldots + r(c_k \tilde{v}_k) \]
\[ = (rc_1) \tilde{v}_1 + (rc_2) \tilde{v}_2 + \ldots + (rc_k) \tilde{v}_k \]
\( \in H \).

(\Rightarrow) If \( \tilde{u} \in H \) and \( r \in \mathbb{R} \) then \( r\tilde{u} \in H \) too.

(\Rightarrow) \( H \) is closed under scalar multiplication.

(\Rightarrow) (a), (b), (c) \( \Rightarrow H \) is a subspace of \( V \).

Example: page 170.

(12) Let \( W \) be the set of all vectors of the form
\[
\begin{bmatrix}
2s + 4t \\
2s \\
2s - 3t \\
0 + 5t
\end{bmatrix}
\]
satisfying

\[
\begin{bmatrix}
2s + 4t \\
2s + 0 \\
2s - 3t \\
0 + 5t
\end{bmatrix} = s
\begin{bmatrix}
2 \\
2 \\
0 \\
0
\end{bmatrix} + t
\begin{bmatrix}
4 \\
0 \\
-3 \\
5
\end{bmatrix}
\]
satisfying

(\Rightarrow) \( W = \text{Span}\left(\begin{bmatrix}2 \\ 2 \\ 0 \\ 1\end{bmatrix}, \begin{bmatrix}4 \\ 0 \\ -3 \\ 5\end{bmatrix}\right)\)

(\Rightarrow) \( W \) is a subspace of \( \mathbb{R}^4 \).