§1.8 Introduction to Linear Transformations

Definition: A transformation, $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, is a function that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ a unique vector denoted $T(\mathbf{x}) \in \mathbb{R}^m$.

$\mathbb{R}^n$ is called the **domain** of $T$, $\mathbb{R}^m$ is called the **co-domain** of $T$.

If $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x}) \in \mathbb{R}^m$ is called the **image** of $\mathbf{x}$ under $T$.

The **range** of $T$ is the set $R = \{ T(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n \}$, the set of all image vectors under $T$.

Diagram:

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\[ \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \]
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**Matrix Transformations**

If $A$ is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$,

Then $AX \in \mathbb{R}^m$, so we can define a transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\mathbf{x}) = AX$.
Example:

Let \( A = \begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & -1 \end{bmatrix} \)

If \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \)

\[
A \cdot x = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \end{bmatrix}
\]

\[
= \begin{bmatrix} x_1 - 2x_2 + 5x_3 \\ 3x_1 - x_3 \end{bmatrix} \in \mathbb{R}^2
\]

\( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) is defined by

\( T(x) = A \cdot x \)

\[
= \begin{bmatrix} x_1 - 2x_2 + 5x_3 \\ 3x_1 - x_3 \end{bmatrix} \in \mathbb{R}^2
\]

\( e.g., T \left( \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2 - 2(1) + 5(-1) \\ 3(2) - (-1) \end{bmatrix} \)

\[
= \begin{bmatrix} 2 - 2 - 5 \\ 6 + 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}
\]
with $T$ defined by $T(x) = Ax$, find a vector $x$ whose image under $T$ is $b$, and determine whether $x$ is unique.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$T(x) = \begin{bmatrix} x_1 \\ -3x_2 \\ -3x_1 + x_2 + 6x_3 \\ 2x_1 - 2x_2 - x_3 \end{bmatrix}$$

To solve $T(x) = b$, we have to solve the system

$$\begin{cases} x_1 - 3x_2 = -2 \\ -3x_1 + x_2 + 6x_3 = 3 \\ 2x_1 - 2x_2 - x_3 = -1 \end{cases}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ -3 & 1 & 6 & 3 \\ 2 & -2 & -1 & -1 \end{bmatrix}$$

$$\text{ref} \left[ \begin{bmatrix} 1 & 0 & -3 & -2 \\ -3 & 1 & 6 & 3 \\ 2 & -2 & -1 & -1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution is

$$x = \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}$$

Since there are no free variables, in the solution set, $x$ is unique.
pg. 669, 9 - 10 Find all \( x \in \mathbb{R}^4 \) that are mapped into the zero vector by the transformation \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \) for the given matrix \( \mathbf{A} \).

\[
\mathbf{A} = \begin{bmatrix}
2 & 2 & 10 & -6 \\
1 & 0 & 2 & -4 \\
0 & 1 & 2 & 3 \\
1 & 4 & 10 & 6
\end{bmatrix}
\]

If \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \)

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix}
3x_1 + 2x_2 + 10x_3 - 6x_4 \\
x_1 + 2x_3 - 4x_4 \\
x_2 + 2x_3 + 3x_4 \\
x_1 + 4x_2 + 10x_3 + 6x_4
\end{bmatrix}
\]

To solve \( T(\mathbf{x}) = \mathbf{0} \), we solve the homogeneous linear system \( \mathbf{A}\mathbf{x} = \mathbf{0} \).

\[
\begin{bmatrix}
3 & 2 & 10 & -6 & | & 0 \\
1 & 0 & 2 & -4 & | & 0 \\
0 & 1 & 2 & 3 & | & 0 \\
1 & 4 & 10 & 6 & | & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & 2 & 4 & | & 0 \\
0 & 1 & 2 & 3 & | & 0 \\
0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]

\[
x_1 = -2x_3 + 4x_4 \\
x_2 = -2x_3 - 3x_4 \\
x_3 = \text{free} \\
x_4 = \text{free}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-2 \\
4 \\
-2 \\
1
\end{bmatrix} + \begin{bmatrix}
x_3 \\
x_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-2 & 4 \\
-2 & -3 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x_3 \\
x_4
\end{bmatrix}
\]
The set of all vectors mapped to 0 by \( X \mapsto A X \) is the set
\[
\text{Span} \left( \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} \right)
\]

Fig. 6. By 13-16 use a Rectangular coordinate system to plot \( \mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \) \( \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \)

and their images under the given transformation \( T \).

\( 14 \) \( T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \)

\[
T \left( \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 10 \\ 4 \end{bmatrix} \quad T \left( \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}
\]
**Linear Transformations**

**Definition:** A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **linear** if and only if:

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$.
2. $T(cu) = cT(u)$ for all $u \in \mathbb{R}^n$, $c \in \mathbb{R}$.

i.e., $T$ is linear if and only if $T$ preserves vector addition and scalar multiplication.

**Note:** If $A$ is an $m \times n$ matrix, $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then by Proposition (§1.4, pg. 30):

1. $A(u + v) = Au + Av$.
2. $A(cu) = cAu$.

Therefore, a matrix transformation is a linear transformation.

**Proposition:** If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then:

1. $T(\overline{0}) = \overline{0}$
2. $T(cu + dv) = cT(u) + dT(v)$ for all $u, v \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

**Proof:**

1. $T(0) = T(u + \overline{0}) = T(u) + T(\overline{0})$.
   $\Rightarrow T(0) - T(u) = T(\overline{0}) + T(0) - T(u)$.
   $\Rightarrow \overline{0} = T(\overline{0}) + \overline{0}$.

Therefore, $T(0) = \overline{0}$. 

*Q.E.D.*
(i) \[ T(c \mathbf{u} + d \mathbf{v}) = T(c \mathbf{u}) + T(d \mathbf{v}) = c \cdot T(\mathbf{u}) + d \cdot T(\mathbf{v}) \]

because \( T \) preserves vector addition and scalar multiplication.

In general - \( T \) preserves linear combinations:

\[ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n \text{ and } c_1, c_2, \ldots, c_k \in \mathbb{R} \]

\[ T(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_k T(\mathbf{v}_k) \]

(2) Let \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) \( \mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \) \( \mathbf{v}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \)

and let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear transformation that maps \( \mathbf{x} \) into \( a \mathbf{v}_1 + b \mathbf{v}_2 \). Find a matrix \( A \) such that

\[ T(\mathbf{x}) = A \mathbf{x} \]

for all \( \mathbf{x} \) in \( \mathbb{R}^2 \)

\[ T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = x_1 \begin{bmatrix} -3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -2 \end{bmatrix} \]

\[ = \begin{bmatrix} -3x_1 + 7x_2 \\ 5x_1 - 2x_2 \end{bmatrix} \]

\[ = \begin{bmatrix} -5 & 7 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ T(\mathbf{x}) = A \mathbf{x} \] for \( A = \begin{bmatrix} -5 & 7 \\ 5 & -2 \end{bmatrix} \).
An affine transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has the form $T(x) = Ax + \overline{b}$ with $A$ an $m \times n$ matrix and $\overline{b}$ a (fixed) vector in $\mathbb{R}^m$.

Show that $T$ is not linear when $\overline{b} \neq \overline{0}$.

If $T(x) = Ax + \overline{b}$ and $\overline{u}, \overline{v} \in \mathbb{R}^n$,

then $T(\overline{u} + \overline{v}) = A(\overline{u} + \overline{v}) + \overline{b} = Au + Av + \overline{b}$

and $T(\overline{u}) + T(\overline{v}) = (A\overline{u} + \overline{b}) + (A\overline{v} + \overline{b}) = Au + Av + 2\overline{b}$

$\Rightarrow T(\overline{u} + \overline{v}) \neq T(\overline{u}) + T(\overline{v})$ if $\overline{b} \neq \overline{0}$.

\[
T(\overline{u} + \overline{v}) - T(\overline{u}) - T(\overline{v}) = \\
Au + Av + \overline{b} - (Au + Av + 2\overline{b}) = \\
Au + Av + \overline{b} - Au - Av - 2\overline{b} = -\overline{b}
\]

if $\overline{b} \neq \overline{0}$.