§ 9.7 Plane Curves and Parametric Equations

**Definition:**

Let \( x = f(t) \) and \( y = g(t) \), where \( f \) and \( g \) are two functions with a common domain, some interval \( I \) of real numbers.

The set of points \( (x, y) = (f(t), g(t)) \) for \( t \) in \( I \) is called a plane curve.

The variable \( t \) is called a parameter and the equations

\[
 x = f(t), \quad y = g(t)
\]

are called parametric equations.

**Example:**

\[
 x = \cos(t), \quad y = \sin(t), \quad 0 \leq t \leq 2\pi
\]

are parametric equations of the unit circle.

\[
 x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1
\]

\( \Rightarrow (x, y) \) is on the unit circle.
Example: If \( a, b > 0 \) and 
\[ x = a \cdot \cos(t) \quad \text{and} \quad y = b \cdot \sin(t) \] 
for \( 0 \leq t \leq 2\pi \), then 
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(t) + \sin^2(t) = 1 \]

\( (x, y) \) is on the ellipse: 
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

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Graph the curve whose parametric equations are given. Find the rectangular equivalent of each curve.

\( \begin{align*} 
\text{(9)} \quad x &= t^2 + 2, \quad y = \sqrt{2t}, \quad t \geq 0 \\
&
\begin{array}{c|c|c|c}
 t & x = t^2 + 2 & y = \sqrt{2t} & (x, y) \\
0 & 2 & 0 & (2, 0) \\
1 & 3 & 1 & (3, 1) \\
4 & 6 & 2 & (6, 2) \\
9 & 11 & 3 & (11, 3) \\
\end{array}
\end{align*} \)

\begin{align*} 
\text{Rectangular equivalent:} \\
y &= \sqrt{x - 2} \quad ; \quad x \geq 2 
\end{align*}
\[ x = 2 \cos(t), \quad y = 3 \sin(t), \quad -\pi \leq t \leq 0 \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x = 2 \cos(t) )</th>
<th>( y = 3 \sin(t) )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\pi)</td>
<td>-2</td>
<td>0</td>
<td>(-2, 0)</td>
</tr>
<tr>
<td>(-\frac{5\pi}{6})</td>
<td>-1.7</td>
<td>-1.5</td>
<td>(-1.7, -1.5)</td>
</tr>
<tr>
<td>(-\frac{3\pi}{4})</td>
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<td>-2.1</td>
<td>(-1.4, -2.1)</td>
</tr>
<tr>
<td>(-\frac{\pi}{3})</td>
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<td>-2.6</td>
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<tr>
<td>(-\frac{\pi}{2})</td>
<td>0</td>
<td>-3</td>
<td>(0, -3)</td>
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<td>1.7</td>
<td>-1.5</td>
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</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>(2, 0)</td>
</tr>
</tbody>
</table>

\[ \frac{x}{2} = \cos(t), \quad \frac{y}{3} = \sin(t) \]

\[ \frac{x^2}{4} + \frac{y^2}{9} = 1 \quad \text{lower half of an ellipse} \]
Find two different parametric equations for the rectangular equation.

31) \( y = x^3 \)
   
   (i) \( x = t \), \( y = t^3 \)
   
   (ii) \( x = 2t \), \( y = 8t^3 \)

32) \( y = x^4 + 1 \)
   
   (i) \( x = t \), \( y = t^4 + 1 \)
   
   (ii) \( x = t - 1 \), \( y = (t-1)^4 + 1 \)