The Ellipse

Theorem: The equation of an ellipse with center at \((0,0)\), and major axis along the x-axis:

with vertices at \((-a,0)\) and \((a,0)\) and foci at \((-c,0)\) and \((c,0)\)

is \[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

where \(a > b\) and \(b^2 = a^2 - c^2\)

For any point \(P(x,y)\) on the ellipse:

\[
\text{dist} (P_1 F_1) + \text{dist} (P_1 F_2) = 2a
\]

If \(P = (a,0)\) then

\[
\text{dist} (P_1 F_1) = \sqrt{b^2+c^2}, \quad \text{dist} (P_1 F_2) = \sqrt{b^2-c^2}
\]

If \(P = (c,0)\)

\[
\text{dist} (P_1 F_1) = a+c, \quad \text{dist} (P_1 F_2) = a-c
\]

\[
\Rightarrow \text{dist} (P_1 F_1) + \text{dist} (P_1 F_2) = (a+c) + (a-c) = 2a
\]

Therefore

\[
\sqrt{b^2+c^2} + \sqrt{b^2-c^2} = 2c \Rightarrow \\
2\sqrt{b^2+c^2} = 2c \Rightarrow \\
\sqrt{b^2+c^2} = c \Rightarrow \\
b^2+c^2 = c^2 \Rightarrow \\
b^2 = c^2 - a^2
\]
Theorem: Equation of an ellipse with center at \((0,0)\) and major axis along the \(y\)-axis.

The vertices are \(V_1 = (0, -a)\) and \(V_2 = (0, a)\), and the foci are \(F_1 = (0, -c)\) and \(F_2 = (0, c)\).

\[
\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1
\]

where \(a > b > 0\) and \(c^2 = a^2 - b^2\).

\[
c^2 + b^2 = a^2 \implies b^2 = a^2 - c^2
\]
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Find the vertices and foci of each ellipse. Graph each equation.

19. \[ \frac{x^2}{9} + \frac{y^2}{25} = 1 \]

\[ \frac{x^2}{1} + \frac{y^2}{9} = 1 \]

\[ b = 3, \quad a = 5, \quad 5 > 3 \]

major axis along the -axis

If the foci are at \((0, -c)\) and \((0, c)\)

\[ c^2 + 3^2 = 5^2 \]

\[ c^2 = 5^2 - 3^2 \]

\[ c^2 = 25 - 9 \]

\[ = 16 \]

\[ \Rightarrow c = 4 \]

The vertices are \((0, 5)\) and \((0, -5)\)

The foci are \((0, 4)\) and \((0, -4)\)
Find an equation for the ellipse.
Graph the equation.

(9) center at (0, 0); focus at (0, -4)
vertex at (0, 5)
vertex on the y-axis
major axis is on the y-axis
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
where \( b = 5 \)
Focus at (0, -4) Therefore \( c = 4 \)

\[ a^2 + 4^2 = c^2 \]
\[ a^2 = 25 - 16 \]
\[ a^2 = 9 \]
\[ a = 3 \]

\[ \frac{x^2}{3^2} + \frac{y^2}{(5)^2} = 1 \]
37. Center at \((0,0)\), vertex at \((0,4)\), \(b=1\)
vertex on \(y\)-axis at \((0,4)\)
major axis along \(y\)-axis, \(a = 4\)

\[
\frac{x^2}{11^2} + \frac{y^2}{4^2} = 1
\]

\[
\frac{x^2}{1} + \frac{y^2}{16} = 1
\]

If the focus is at \((0,1)\)

\[
c^2 + 1^2 = (4)^2 \Rightarrow c^2 + 1 = 16 \Rightarrow c^2 = 15 \Rightarrow c = \sqrt{15}
\]

The foci are at \((0,1-\sqrt{15})\), \((0,1+\sqrt{15})\)