8.4.1 Vectors

A vector is a quantity that has both magnitude and direction.

A vector is represented by a directed line segment $\overrightarrow{PQ}$, with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$.

The $x$-component of the vector represented by $\overrightarrow{PQ}$ is $x_2 - x_1$.

The $y$-component of the vector represented by $\overrightarrow{PQ}$ is $y_2 - y_1$.

The magnitude of the vector represented by $\overrightarrow{PQ}$ is the length of the directed line segment:

$$|\overrightarrow{PQ}| = \text{dist}(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The direction of the vector represented by $\overrightarrow{PQ}$ is from $P$ to $Q$.

Two different directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ represent the same vector if and only if they have the same magnitude and the same direction.
The zero vector \( \vec{0} \) is the vector with zero magnitude and no direction. \( \vec{0} = \overrightarrow{PP} \) (initial point and terminal point are the same.)

Adding vectors geometrically:
If \( \overrightarrow{V} \) and \( \overrightarrow{W} \) are two vectors, we define the sum \( \overrightarrow{V} + \overrightarrow{W} \) of \( \overrightarrow{V} \) and \( \overrightarrow{W} \) as follows:

Position \( \overrightarrow{V} \) and \( \overrightarrow{W} \) so that the initial point of \( \overrightarrow{W} \) coincides with the terminal point of \( \overrightarrow{V} \), then \( \overrightarrow{V} + \overrightarrow{W} \) is the unique vector whose initial point coincides with the initial point of \( \overrightarrow{V} \) and whose terminal point coincides with the terminal point of \( \overrightarrow{W} \).

[Diagram showing \( \overrightarrow{V} + \overrightarrow{W} \) with \( \overrightarrow{V} \) and \( \overrightarrow{W} \) illustrated]

**Algebraic properties of vector addition**
- If \( \overrightarrow{U}, \overrightarrow{V}, \) and \( \overrightarrow{W} \) are vectors in the plane
  - (i) \( \overrightarrow{V} + \overrightarrow{W} = \overrightarrow{W} + \overrightarrow{V} \) (commutative law)
  - (ii) \( \overrightarrow{U} + (\overrightarrow{V} + \overrightarrow{W}) = (\overrightarrow{U} + \overrightarrow{V}) + \overrightarrow{W} \) (associative law)
  - (iii) \( \overrightarrow{V} + \overrightarrow{0} = \overrightarrow{V} \) (identity property)

If \( \overrightarrow{V} \) is a vector, \( -\overrightarrow{V} \) is the vector that has the same magnitude as \( \overrightarrow{V} \) and faces the opposite direction as \( \overrightarrow{V} \).

i.e., if \( \overrightarrow{V} = \overrightarrow{PQ} \) then \( -\overrightarrow{V} = \overrightarrow{QP} \)

Then
- (iv) \( \overrightarrow{V} + (-\overrightarrow{V}) = \overrightarrow{0} \) (inverse property)
Vector Subtraction:
If \( \overrightarrow{V} \) and \( \overrightarrow{W} \) are vectors, the difference \( \overrightarrow{V} - \overrightarrow{W} \) is defined by
\[
\overrightarrow{V} - \overrightarrow{W} = \overrightarrow{V} + (-\overrightarrow{W})
\]

Scalar Multiplication:
A number is called a scalar when we are dealing with vectors.
If \( \alpha \) is a scalar and \( \overrightarrow{V} \) is a vector,
(i) If \( \alpha > 0 \), then \( \alpha \overrightarrow{V} \) is the vector whose magnitude is \( \alpha \) times the magnitude of \( \overrightarrow{V} \) and whose direction is the same as the direction of \( \overrightarrow{V} \).
\[
\|\alpha \overrightarrow{V}\| = \alpha \|\overrightarrow{V}\|
\]
(ii) If \( \alpha < 0 \), \( \alpha \overrightarrow{V} \) is the vector whose magnitude is \( |\alpha| \) times the magnitude of \( \overrightarrow{V} \) and whose direction is the opposite of the direction of \( \overrightarrow{V} \).
\[
\|\alpha \overrightarrow{V}\| = |\alpha| \|\overrightarrow{V}\|
\]
\[
\alpha \overrightarrow{V} = |\alpha| (-\overrightarrow{V})
\]
(ii) If \( a = 0 \) or \( \vec{V} = \vec{0} \) then \( a \vec{V} = \vec{0} \).

**Properties:**

(i) \( 0 \vec{V} = \vec{0} \)

(ii) \( 1 \vec{V} = \vec{V} \)

(iii) \( -1 \vec{V} = -\vec{V} \)

(iv) \( (\alpha + \beta) \vec{V} = \alpha \vec{V} + \beta \vec{V} \)

(v) \( a(\vec{V} + \vec{W}) = a\vec{V} + a\vec{W} \)

(vi) \( a(\beta \vec{V}) = (a \beta) \vec{V} \)

**Graphing vectors**

\( \vec{W} \) has components \([2, 5]\)

\( \vec{U} \) has components \([2, -1]\)

\( \vec{V} \) has components \([5, 0]\)

**Graph:**

(9) \( \vec{V} + \vec{W} \)

\( \vec{V} + \vec{W} \) has components \([7, 5]\)

\([5, 0] + [2, 5] = [5 + 2, 0 + 5] = [7, 5]\)
13. \( \vec{V} - \vec{W} \)
\( \vec{V} \) has components \([5, 0]\)
\( -\vec{W} \) has components \([-2, -5]\)

\[ \vec{V} - \vec{W} \text{ has components } [3, -5] \]
\[ [5, 0] - [2, 5] = [3, -5] \]
\[ [5, 0] + [-2, -5] = [3, -5] \]

11. \( 3\vec{V} \)
\( \vec{V} \) has components \([5, 0]\)

\( 3\vec{V} \) has components \([15, 0]\)
\[8.4.6\]

If \( \overrightarrow{V} = \langle a, b \rangle \) is an algebraic vector whose initial point is at \((0,0)\), the origin, then \( \overrightarrow{V} \) is called a position vector.

**Theorem:** Suppose \( \overrightarrow{V} \) is a vector represented by the directed line segment \( PQ \) where \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \). Then \( \overrightarrow{V} \) is equal to the position vector \( \overrightarrow{V} = \langle x_2 - x_1, y_2 - y_1 \rangle \).

If \( a = x_2 - x_1 \), \( b = y_2 - y_1 \),

\[ \overrightarrow{V} \text{ and } \overrightarrow{PQ} \text{ have the same direction and the same magnitude, so they are equal.} \]

**Example:** If \( \overrightarrow{V} = \overrightarrow{PQ} \) where \( P = (-2, 0) \) and \( Q = (3, 4) \),

Then \( \overrightarrow{V} = \langle 3 - (-2), 4 - 0 \rangle = \langle 5, 4 \rangle \)

is the position vector for \( \overrightarrow{V} \).
Equality of Vectors
If \( \vec{V} = \langle a_1, b_1 \rangle \) and \( \vec{W} = \langle a_2, b_2 \rangle \)
then \( \vec{V} = \vec{W} \) if and only if
\[ a_1 = a_2 \quad \text{and} \quad b_1 = b_2 \]

Standard Unit Vectors
- The standard unit vector in the x-direction is \( \hat{i} = \langle 1, 0 \rangle \)
- The standard unit vector in the y-direction is \( \hat{j} = \langle 0, 1 \rangle \)

If \( \vec{V} = \langle a_1, b_1 \rangle \) then
\[ \vec{V} = \langle a_1, 0 \rangle + \langle 0, b_1 \rangle \]
\[ = a_1 \hat{i} + b_1 \hat{j} \]

Adding and Subtracting Vectors
Algebraically
If \( \vec{V} = \langle a_1, b_1 \rangle = a_1 \hat{i} + b_1 \hat{j} \)
\( \vec{W} = \langle a_2, b_2 \rangle = a_2 \hat{i} + b_2 \hat{j} \)
then
\[ \vec{V} + \vec{W} = (a_1 + a_2) \hat{i} + (b_1 + b_2) \hat{j} \]
\[ = \langle a_1 + a_2, b_1 + b_2 \rangle \]

and
\[ \vec{V} - \vec{W} = (a_1 - a_2) \hat{i} + (b_1 - b_2) \hat{j} \]
\[ = \langle a_1 - a_2, b_1 - b_2 \rangle \]

If \( \alpha \) is a scalar
\[ \alpha \vec{V} = (\alpha a_1) \hat{i} + (\alpha b_1) \hat{j} \]
\[ = \langle \alpha a_1, \alpha b_1 \rangle \]

1. \( \| \vec{V} \| = \sqrt{a_1^2 + b_1^2} \)
The vector $\mathbf{v}$ has initial point $P$ and terminal point $Q$. Write $\mathbf{v}$ in the form $a\mathbf{i} + b\mathbf{j}$. Find its position vector.

30. $P = (-3, 2), \quad Q = (6, 5)$

$\mathbf{v} = (6 - (-3), 5 - 2) = (9, 3)$

$= 9\mathbf{i} + 3\mathbf{j}$

Find $|\mathbf{v}|$

36. $\mathbf{v} = -5\mathbf{i} + 12\mathbf{j}$

$|\mathbf{v}| = \sqrt{(-5)^2 + (12)^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$

If $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} = <3, -5>$, $\mathbf{w} = -2\mathbf{i} + 3\mathbf{j} = <-2, 3>$

Find

42. $3\mathbf{v} - 2\mathbf{w} = 3<3, -5> - 2<-2, 3>$

$= <9, -15> + <4, -6>$

$= <13, -21>$

$= 13\mathbf{i} - 21\mathbf{j}$

43. $|\mathbf{v} - \mathbf{w}|$

$\mathbf{v} - \mathbf{w} = <3, -5> - <-2, 3>$

$= <3 - (-2), -5 - 3>$

$= <5, -8>$

$|\mathbf{v} - \mathbf{w}| = \sqrt{(5)^2 + (-8)^2}$

$= \sqrt{25 + 64}$

$= \sqrt{89} \\ \approx 9.43$ units
8. 4. 9

- Finding a unit vector

If \( \vec{V} = <a, b> \neq <0, 0> \)

\[ ||\vec{V}|| = \sqrt{a^2 + b^2} \neq 0 \]

The unit vector in the direction of \( \vec{V} \)

is \( \vec{U} = \frac{1}{||\vec{V}||} \vec{V} \)

\[ = \left( \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right) \]

Example: pg. 628

Find the unit vector in the direction of \( \vec{V} \).

\[ \vec{V} = 3\hat{i} - 4\hat{j} = <3, -4> \]

\[ ||\vec{V}|| = \sqrt{3^2 + (-4)^2} \]

\[ = \sqrt{9 + 16} \]

\[ = \sqrt{25} \]

\[ = 5 \]

\[ \vec{U} = \frac{1}{5} \vec{V} \]

\[ = \frac{1}{5} <3, -4> \]

\[ = \left( \frac{3}{5}, -\frac{4}{5} \right) \]

\[ = \left( \frac{3\hat{i}}{5} - \frac{4\hat{j}}{5} \right) \]

- Finding a vector from its direction and magnitude

If \( \vec{V} = <a, b> \)

its direction angle \( \theta \) is the angle between the positive x-axis and the vector \( \vec{V} \)

\[ \tan(\theta) = \frac{b}{a} \quad \text{if} \ a \neq 0 \]
The unit vector in the direction of \( \vec{V} \) is
\[
\vec{U} = \frac{\vec{V}}{||\vec{V}||} \Rightarrow \vec{V} = ||\vec{V}|| \vec{U}
\]
\[
= ||\vec{V}|| < \cos(\theta), \sin(\theta) >=
\]
\[
= ||\vec{V}|| < \cos(\theta), \sin(\theta) >
\]
\[
= \left( ||\vec{V}|| \cos(\theta), ||\vec{V}|| \sin(\theta) \right)
\]

I.e.
If the magnitude of \( \vec{V} \) is \( ||\vec{V}|| \)
and the direction angle of \( \vec{V} \) is \( \theta \)
then
\[
\vec{V} = ||\vec{V}|| \left( \cos(\theta) \hat{i} + \sin(\theta) \hat{j} \right)
\]

Example: pg. 62
Write the vector \( \vec{V} \) in the form \( \vec{V} = a\hat{i} + b\hat{j} \)
given its magnitude \( ||\vec{V}|| \) and its direction angle \( \alpha \).

(6) \( ||\vec{V}|| = 5, \alpha = 45^\circ \)
\[
\vec{V} = (5 \cos(45^\circ), 5 \sin(45^\circ)) = (5 \left( \frac{\sqrt{2}}{2} \right), 5 \left( \frac{\sqrt{2}}{2} \right)) = (4.12, 4.12)
\]
\[
= 4.12 \hat{i} + 4.12 \hat{j}
\]

(6) \( ||\vec{V}|| = 25, \alpha = 330^\circ \) (in QIV \( 330^\circ = -30^\circ + 360^\circ \))
\[
\vec{V} = (25 \cos(330^\circ), 25 \sin(330^\circ)) = (25 \left( \frac{\sqrt{3}}{2} \right), 25 \left( -\frac{1}{2} \right)) = (12.5 \sqrt{3}, -12.5)
\]
\[
= (12.5 \sqrt{3}) \hat{i} - (12.5) \hat{j}
\]
Find the direction angle between \( \vec{v} \) and \( \vec{v} \) for each vector.

\[
\vec{V} = 4\hat{i} - 2\hat{j} = \langle 4, -2 \rangle \quad \text{in } \mathbb{QIV}
\]

\[
\alpha = \tan^{-1}\left(\frac{-2}{4}\right) = \tan^{-1}\left(-\frac{1}{2}\right)
\]

\[
\alpha = -26.57^\circ
\]

or \[-26.57^\circ + 360^\circ = 333.43^\circ\]

\[
\|\vec{V}\| = \sqrt{(4)^2 + (-2)^2}
\]

\[
= \sqrt{16 + 4}
\]

\[
= \sqrt{20}
\]

\[
= 2\sqrt{5}
\]

\[
\vec{V} = 2\sqrt{5} \left( \cos(333.43^\circ) \hat{i} + \sin(333.43^\circ) \hat{j} \right)
\]

MODEL WITH VECTORS

Forces can be represented by vectors because they have both magnitude and direction. If two force vectors \( \vec{F}_1 \) and \( \vec{F}_2 \) act on an object at the same time, the net effect experienced by the object is called the resultant force. The resultant force is equal to the vector sum \( \vec{F}_1 + \vec{F}_2 \).
Example: page 629

Resultant Force

Two forces of magnitude 40 newtons (N) and 60N act on an object at angles of 30°
and 45° with respect to the positive x-axis. Find the direction and the magnitude of
the resultant force.

\[ \text{Resultant force} \]

\[ \vec{F}_{\text{resultant}} = 40N \]

\[ \vec{F}_1 = <40N \cos(30°), 40N \sin(30°)> \]
\[ = <40N \left( \frac{\sqrt{3}}{2} \right), 40N \left( \frac{1}{2} \right)> \]
\[ = <20\sqrt{3}N, 20N> \]

\[ \vec{F}_2 = <60N \cos(-45°), 60N \sin(-45°)> \]
\[ = <60N \left( \frac{\sqrt{2}}{2} \right), 60N \left( -\frac{\sqrt{2}}{2} \right)> \]
\[ = <30\sqrt{2}N, -(30\sqrt{2})N> \]

The resultant force is

\[ \vec{F}_1 + \vec{F}_2 = <20\sqrt{3}N + 30\sqrt{2}N, 20N - 30\sqrt{2}N> \]
\[ \approx <77.07N, -22.43N> \]

Magnitude

\[ || \vec{F}_1 + \vec{F}_2 || \approx \sqrt{(77.07)^2 + (-22.43)^2} \]
\[ \approx 80.27N \]

Direction angles

\[ \tan(\theta) = -\frac{22.43}{77.07} \text{ in QIII} \]

\[ \theta = \tan^{-1} \left( \frac{-22.43}{77.07} \right) \]
\[ \approx -16.22° \]

80.27N in the direction -16.22° w.r.t. to the positive x-axis.