§8.2 Polar Equations and Graphs

1. Graph and identify polar equations by converting to rectangular coordinates
2. Test polar equations for symmetry
3. Graph polar equations by plotting points

Examples: page 667
Transform each polar equation to an equation in rectangular coordinates. Then identify and graph the equation.

(13) $r = 4 \implies r^2 = 16$
     $\implies x^2 + y^2 = 16$
a circle of radius $r = 4$ with center at (0,0)

(16) $\theta = \frac{\pi}{3} \implies \tan(\theta) = \sqrt{3}$
     $\implies \frac{y}{x} = \sqrt{3}$
     $\implies y = \sqrt{3} \cdot x$
a line through (0,6) with slope $m = \sqrt{3}$
17) \( r \sin(\theta) = 4 \implies y = 4 \)

A horizontal line with \( y \)-intercept \((0, 4)\)

\[ y = 4 \quad (r \sin(\theta) = 4) \]

16) \( r \cdot \cos(\theta) = -2 \implies x = -2 \)

A vertical line with \( x \)-intercept \((-2, 0)\)

\[ x = -2 \quad (r \cdot \cos(\theta) = -2) \]

21) \( r = 2 \cos(\theta) \)

\[ r^2 = 2r \cos(\theta) \implies x^2 + y^2 = 2x \implies \]

\[ (x^2 - 2x) + y^2 = 0 \implies (x^2 - 2x + 1) + y^2 = 1 \implies (x - 1)^2 + y^2 = 1 \]

A circle of radius 1 and with center at the point \((1, 0)\)

\[ r = 2 \cdot \cos(\theta) \]

\[ (x - 1)^2 + y^2 = 1 \]
27) \( r \cdot \csc(\theta) = -2 \)

\[ \frac{r}{\sin(\theta)} = -2 \]

\[ R = -2 \sin(\theta) \Rightarrow \]

\[ r^2 = -2r \sin(\theta) \Rightarrow \]

\[ x^2 + y^2 = -2y \Rightarrow \]

\[ x^2 + y^2 - 2y = 0 \Rightarrow \]

\[ x^2 + (y^2 - 2y + 1) = 1 \Rightarrow \]

\[ x^2 + (y - 1)^2 = 1 \]

A circle of radius 1 and center at the point \((0,1)\)

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**Theorem:** Let \( a \) be a non-zero real number, then the graph of the equation

\[ r \cdot \sin(\theta) = a \]

is equivalent to \( y = a \)

A horizontal line with \( y \)-intercept \((0,a)\)

The graph of the equation

\[ r \cdot \cos(\theta) = a \]

is equivalent to \( x = a \)

A vertical line with \( x \)-intercept \((a,0)\)
Theorem: Let \( a \) be a positive real number.

Then:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 2a \sin(\theta) )</td>
<td>Circle: radius ( a ), center ((0,0)) in rectangular coordinates</td>
</tr>
<tr>
<td>( r = -2a \sin(\theta) )</td>
<td>Circle: radius ( a ), center ((-a, 0)) in rectangular coordinates</td>
</tr>
<tr>
<td>( r = 2a \cos(\theta) )</td>
<td>Circle: radius ( a ), center ((a, 0)) in rectangular coordinates</td>
</tr>
<tr>
<td>( r = -2a \cos(\theta) )</td>
<td>Circle: radius ( a ), center ((-a, 0)) in rectangular coordinates</td>
</tr>
</tbody>
</table>

Each circle passes through the origin.

Tests For Symmetry:

Theorem:

- **Symmetry with respect to \( x\)-axis.**
  - In a polar graph, replace \( \theta \) with \(-\theta\).
  If an equivalent equation results, the graph is symmetric with respect to the \( x\)-axis. (The line \( \theta = 0 \))

- **Symmetry with respect to the \( y\)-axis.**
  - In a polar graph, replace \( \theta \) by \( \pi - \theta \).
  If an equivalent equation results, the graph is symmetric with respect to the \( y\)-axis. (The line \( \theta = \frac{\pi}{2} \))

- **Symmetry with respect to the origin.**
  - In a polar equation, replace \( r \) by \(-r\) or replace \( \theta \) with \( \theta + \pi \).
  If an equivalent equation results, the graph is symmetric with respect to the origin \((r = 0)\).
Symmetry with respect to the x-axis:

Symmetry with respect to the y-axis:

\[-(r, \theta) = (r, \theta + \pi)\]
Examples:

- \( r = 2 \cdot \cos(\theta) \)
  
  symmetry w.r.t. x-axis
  
  (x-axis) \( 2 \cdot \cos(-\theta) = 2 \cdot \cos(\theta) \)
  
  because cosine is an even function
  
  The graph is symmetric about the x-axis
  
  (y-axis) \( 2 \cdot \cos(\pi - \theta) = 2 \cdot (-\cos(\theta)) = -2 \cdot \cos(\theta) \)
  
  \( \cos(\pi - \theta) = \cos(\pi) \cdot \cos(\theta) + \sin(\pi) \cdot \sin(\theta) \)
  
  = \((-1) \cdot \cos(\theta) + (0) \cdot \sin(\theta) \)
  
  = \(-\cos(\theta) \)
  
  The graph is not symmetric about the y-axis.
  
  (origin) if \( (-r) = 2 \cdot \cos(\theta) \)
  then \( r = -2 \cdot \cos(\theta) \neq 2 \cdot \cos(\theta) \)
  
  The graph is not symmetric about the origin.

- \( r^2 = 4 \sin(2\theta) \)
  
  (x-axis) replace \( \theta \) with \(-\theta \)
  
  \( 4 \sin(2(-\theta)) = 4 \sin(-2\theta) \)
  
  = \(-4 \sin(2\theta) \)
  
  \( \neq 4 \sin(2\theta) \)
  
  If \( r^2 = 4 \sin(-2\theta) \)
  
  \( r^2 = -4 \sin(2\theta) \)
  
  \( \neq 4 \sin(2\theta) \)
  
  The graph is not symmetric about the x-axis.
\[ r^2 = 4 \sin(2\theta) \]

(y-axis) replace \( \theta \) with \( \pi - \theta \)

\[ 4 \sin(2(\pi - \theta)) = \]
\[ 4 \sin(2\pi - 2\theta) = \]
\[ 4 \sin(-2\theta) = \]
\[ -4 \sin(2\theta) = \neq 4 \sin(2\theta) \]

The graph is not symmetric about the y-axis.

(origin) replace \( r \) with \( -r \)

\[ (-r)^2 = r^2 \]
\[ (-r)^2 = 4 \sin(2\theta) \] is equivalent to \( r^2 = 4 \sin(2\theta) \)

The graph is symmetric about the origin. (see Figure 31 pg. 404)

\section*{Graphing Polar Equations by Plotting Points}

\[ r = \frac{2}{1 - \cos(\theta)} \text{ (parabola)} \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r )</th>
<th>( \theta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>undefined</td>
<td>( \pm \frac{\pi}{6} )</td>
<td>1.07</td>
</tr>
<tr>
<td>( \pm \frac{\pi}{4} )</td>
<td>14.93</td>
<td>( \pm \frac{4\pi}{3} )</td>
<td>1.33</td>
</tr>
<tr>
<td>( \pm \frac{3\pi}{2} )</td>
<td>4</td>
<td>( \frac{3\pi}{2} )</td>
<td>2</td>
</tr>
<tr>
<td>( \pm \frac{5\pi}{3} )</td>
<td>1.33</td>
<td>( \frac{5\pi}{3} )</td>
<td>4</td>
</tr>
<tr>
<td>( \pm \frac{11\pi}{6} )</td>
<td>1.07</td>
<td>( 2\pi )</td>
<td>undefined</td>
</tr>
</tbody>
</table>
\[ r = \frac{2}{1 - \cos(\theta)} \]
73 \quad r = \frac{1}{3 - 2 \cos(\theta)} \quad (ellipse)

\begin{array}{c|c}
\theta & r \\
0 & 1.00 \\
\frac{\pi}{6} & 0.79 \\
\frac{\pi}{3} & 0.50 \\
\frac{\pi}{2} & 0.33 \\
\frac{2\pi}{3} & 0.25 \\
\frac{5\pi}{6} & 0.21 \\
\pi & 0.20 \\
\frac{7\pi}{6} & 0.21 \\
\frac{2\pi}{3} & 0.25 \\
\frac{5\pi}{4} & 0.33 \\
\frac{7\pi}{4} & 0.50 \\
\frac{9\pi}{4} & 0.70 \\
2\pi & 1.00 \\
\end{array}
Example:

$$r = 1 + \sin(\theta) \quad \text{(Cardioid)}$$

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
</tr>
<tr>
<td>(\pi / 6)</td>
<td>1.50</td>
</tr>
<tr>
<td>(\pi / 3)</td>
<td>1.71</td>
</tr>
<tr>
<td>(\pi / 2)</td>
<td>1.87</td>
</tr>
<tr>
<td>(2\pi / 3)</td>
<td>2.00</td>
</tr>
<tr>
<td>(5\pi / 6)</td>
<td>1.87</td>
</tr>
<tr>
<td>(\pi)</td>
<td>1.71</td>
</tr>
<tr>
<td>3\pi / 2</td>
<td>1.50</td>
</tr>
<tr>
<td>2\pi</td>
<td>1.00</td>
</tr>
</tbody>
</table>
**Cardioids**

Cardioids are characterized by equations of the form:

1. \( r = a(1 + \cos(\theta)) \)  symmetric about the \( x \)-axis
2. \( r = a(1 - \cos(\theta)) \)  symmetric about the \( x \)-axis
3. \( r = a(1 + \sin(\theta)) \)  symmetric about the \( y \)-axis
4. \( r = a(1 - \sin(\theta)) \)  symmetric about the \( y \)-axis

**Examples**

1. \( r = 2(1 + \cos(\theta)) \)

2. \( r = 2(1 - \cos(\theta)) \)

3. \( r = 2(1 + \sin(\theta)) \)
\[ r = 2(1 - \sin(\theta)) \]

\[ r = \cos(\theta) \]

\[ r = \cos(2\theta) \]

\[ r = \cos(3\theta) \]