§5.1.1 Angles and their Measure

Definitions:
- A ray is a half-line starting at a point, V, and extending indefinitely in one direction. V is called the vertex of the ray.

- Two rays drawn with a common vertex form an angle. We call one ray the initial side of the angle, and the other side the terminal side of the angle.

- The angle formed is identified by showing the direction and the amount of rotation from the initial side to the terminal side.

- If the rotation is in the counterclockwise direction the angle is positive.

- If the rotation is in the clockwise direction the angle is negative.

- Angles are usually denoted by lower case Greek letters such as \( \alpha, \beta, \gamma, \delta, \Theta, \phi \).
Examples:

- **Terminal side**
- **Initial side**
- **Positive angle**
- **Negative angle**
- **Positive angle more than one full rotation**
- **Negative angle more than one full rotation**
- **Standard Position of an Angle.**

  An angle, \( \theta \), is said to be in standard position if and only if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive x-axis.

  Example:

  ![Standard Position of an Angle](image)

- When an angle is in standard position, its terminal side will either lie in one of the four quadrants, in which case we say the angle is in that quadrant, or the terminal side of the angle will lie on the x-axis or on the y-axis, in which case we say the angle is a quadrantal angle.

  Example: \( \theta \) is in quadrant III.
Example:

\[ \theta \text{ is a quadrantal angle.} \]

\[ \theta \]

- The measure of an angle
  - We use two different systems for measuring angles. The systems are degrees and radians.
    - Degrees
    - In the degree system, the angle that corresponds to one full rotation in the counter-clockwise direction is 360°.
    - One full rotation in the clockwise direction is -360°.

- \(-1^\circ\) is \(\frac{1}{360}\) revolution in the counter-clockwise direction.
A rotation counter-clockwise is $90^\circ$.

A rotation counter-clockwise is $180^\circ$.

A rotation counter-clockwise is $270^\circ$.

A rotation counter-clockwise is $360^\circ$.

A rotation counter-clockwise is $360^\circ$.

A rotation counter-clockwise is $360^\circ$.

$45^\circ$ is $\frac{1}{2}$ of $90^\circ$.

$36^\circ$ is $\frac{1}{3}$ of $90^\circ$. 
Draw the angle:

12. $60^\circ$

$60^\circ$ is $\frac{1}{3}$ of $180^\circ$ or $\frac{2}{3}$ of $90^\circ$

14. $-120^\circ$

$-120^\circ$ is 2 times $-60^\circ$ or $\frac{2}{3}$ times $-180^\circ$

16. $540^\circ$

$540^\circ$ is $180^\circ + 3(60^\circ)$
Converting between Decimal Degrees and Degrees, minutes, seconds.

Subdivisions of degrees may be obtained using decimal degrees, but we also use subdivisions involving minutes and seconds.

- One minute, denoted by $1'$, is equal to $\frac{1}{60}$ of a degree.

- One second, denoted by $1''$, is equal to $\frac{1}{60}$ minute or $\frac{1}{3600}$ of a degree.

**Examples:**

An angle of 65 degrees 20 minutes 16 seconds is denoted by $65^\circ 20' 16''$.

Converting from DMS to D.D.

Example:

$65^\circ 20' 16'' = 65 + \frac{20}{60} + \frac{16}{3600}$

$\approx 65.3378^\circ$

**Using T 1 - 89**

(2nd 1) (2nd 2) (2nd 4)

$65^\circ 20' 16'' \rightarrow$ D.D.

(2nd MATH) (2: Angle) (9: DDD)

**Casio:**

Action/Transform/dms

dms (65, 20, 16)

$65.3378$
Converting from D.D. to D.M.S.

**Example:** Convert $25.456^\circ$ to DMS

$25.456^\circ = 25^\circ + 0.456^\circ$

$= 25^\circ + 0.456(60')$

$= 25^\circ + 27.36'$

$= 25^\circ 27' + 0.36''$

$= 25^\circ 27' + 0.36(60'')$

$= 25^\circ 27' + 21.6''$

$\approx 25^\circ 27' 22''$

TI-89

$25.456^\circ \rightarrow \text{DMS}$

(2nd Math) (2: Angle) (8: DMS)

$= 25^\circ 27' 21.6''$

- Casio: Action/Transform $\rightarrow \text{DMS} (25.456)$
- **Radian Measure**

**Definition:** A central angle in a circle of radius $r$ is an angle whose vertex is at the center of the circle.

The sides of the angle subtend an arc on the circumference of the circle.

A central angle in a circle of radius $r$ that subtends an arc also of length $r$ has a radian measure of $1$ (radian).
**Definition:** If a central angle in a circle of radius \( r \) subtends an arc of length \( s \), the radian measure of the angle is

\[
\theta = \frac{s}{r} \quad \text{(Note: Units cancel)}
\]

**Note:**

The circumference of a circle of radius \( r \) is \( 2\pi r \), so the radian measure of one full rotation in the counter-clockwise direction is

\[
\theta = \frac{2\pi r}{r} = 2\pi \text{ (radians)}
\]

**Examples:**

- \( 360^\circ = 2\pi \text{ (radians)} \)
- \( 180^\circ = \pi \)
- \( 90^\circ = \frac{\pi}{2} \)
- \( 60^\circ = \frac{\pi}{3} \)
- \( 45^\circ = \frac{\pi}{4} \)
- \( 30^\circ = \frac{\pi}{6} \)
- \( 0^\circ = 0 \)
Find the length of the arc of a circle.

\[
\theta = \frac{s}{r} \Rightarrow s = r \cdot \theta
\]

A central angle of radian measure \( \theta \) in a circle of radius \( r \) subtends an arc of length \( s \) where \( s = r \cdot \theta \)

Example: page 388

72. \( r = 6 \) feet, \( \theta = 2 \) (rad), \( s = ? \)
\[
s = r \cdot \theta \\
= (6 \text{ feet}) \cdot (2) \\
= 12 \text{ feet}
\]

In a circle of radius 6 feet, an angle of radian measure 2 subtends an arc of length 12 feet.

76. \( r = 6 \) meters, \( s = 8 \) meters, \( \theta = ? \)
\[
\theta = \frac{s}{r} \\
= \frac{8 \text{ meters}}{6 \text{ meters}} \\
= \frac{4}{3} \text{ (rad)}
\]
Converting from degrees to radians:

\[360^\circ = 2\pi \text{ (rad.)}\]
\[180^\circ = \pi \text{ (rad.)}\]
\[1^\circ = \frac{\pi}{180} \text{ (rad.)}\]

Or

\[x^\circ = x \left(\frac{\pi}{180}\right) \text{ (rad.)}\]

Example:

\[225^\circ = 225 \left(\frac{\pi}{180}\right) \text{ (rad.)}\]
\[-= \left(\frac{225}{180}\right) \pi \text{ (rad.)}\]
\[= \frac{5}{4} \pi \text{ (rad.)}\]

Converting from radians to degrees:

\[\pi \text{ (rad.)} = 180^\circ\]
\[1 \text{ (rad.)} = \left(\frac{180}{\pi}\right)^\circ\]

Or

\[x \text{ (rad.)} = x \left(\frac{180}{\pi}\right)^\circ\]

Example:

\[\frac{5\pi}{6} \text{ (rad.)} = \frac{5\pi}{6} \left(\frac{180}{\pi}\right)^\circ\]
\[= \frac{5}{6} \left(\frac{180}{\pi}\right)^\circ\]
\[= 5 \left(\frac{360}{\pi}\right)^\circ\]
\[= 150^\circ\]
Example: convert 110 degrees

\[ 5 \pi - 3\frac{\pi}{4} = -3\frac{\pi}{4} \left( \frac{180}{\pi} \right) \]

\[ = -3(45) \]

\[ = -135^\circ \]

Table 1

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>0(^\circ)</td>
<td>0</td>
</tr>
<tr>
<td>30(^\circ)</td>
<td>(\pi/6)</td>
</tr>
<tr>
<td>45(^\circ)</td>
<td>(\pi/4)</td>
</tr>
<tr>
<td>60(^\circ)</td>
<td>(\pi/3)</td>
</tr>
<tr>
<td>90(^\circ)</td>
<td>(\pi/2)</td>
</tr>
<tr>
<td>120(^\circ)</td>
<td>(2\pi/3)</td>
</tr>
<tr>
<td>135(^\circ)</td>
<td>(3\pi/4)</td>
</tr>
<tr>
<td>150(^\circ)</td>
<td>(5\pi/6)</td>
</tr>
<tr>
<td>180(^\circ)</td>
<td>(\pi)</td>
</tr>
<tr>
<td>210(^\circ)</td>
<td>(7\pi/6)</td>
</tr>
<tr>
<td>225(^\circ)</td>
<td>(5\pi/4)</td>
</tr>
<tr>
<td>240(^\circ)</td>
<td>(4\pi/3)</td>
</tr>
<tr>
<td>270(^\circ)</td>
<td>(3\pi/2)</td>
</tr>
<tr>
<td>300(^\circ)</td>
<td>(5\pi/3)</td>
</tr>
<tr>
<td>315(^\circ)</td>
<td>(7\pi/4)</td>
</tr>
<tr>
<td>330(^\circ)</td>
<td>(11\pi/6)</td>
</tr>
<tr>
<td>360(^\circ)</td>
<td>(2\pi)</td>
</tr>
</tbody>
</table>
Find the area of a sector of a circle.

Let \( \theta \) be a central angle in a circle of radius \( r \), where \( \theta \) is measured in radians. We want to find the area of the sector of the circle determined by \( \theta \).

Let \( A \) be the area of the sector. The area bounded by the circle is \( \pi r^2 \). The angle corresponding to one full rotation is \( 2\pi \) (radians) by proportionality:

\[
\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \Rightarrow A = \frac{\theta}{2\pi} (\pi r^2)
\]

or

\[
A = \frac{1}{2} r^2 \theta
\]
A central angle $\theta$, measured in radians, in a circle of radius $r$ determines a sector of area

$$A = \frac{1}{2} r^2 \theta$$

**Example:** page 388

(80) $r = 6$ feet, $\theta = 2$ (rad), $A = \frac{2}{2} r^2 \theta$

$$A = \frac{1}{2} (6 \text{ feet})^2 (2)$$

$$= \frac{1}{2} (36 \text{ feet}^2 (2)$$

$$= 36 \text{ ft}^2$$

(84) $r = 6$ meters, $A = 8 \text{(meters)}^2$, $\theta = \frac{2}{2}$

$$A = \frac{1}{2} (r^2) \theta$$

$$8 \text{ m}^2 = \frac{1}{2} (r \text{ cm})^2 \theta$$

$$8 \text{ m}^2 = \frac{1}{2} (36 \text{ cm})^2 \theta$$

$$8 \text{ m}^2 = 18 \text{ m}^2 \theta$$

$$\theta = \frac{8 \text{ m}^2}{18 \text{ m}^2}$$

$$\theta = \frac{4}{9} \text{(rad.)}$$