In Units 7 – 9 we will study the different ways that we can transform the graph of one function in order to create the graph of another function. We can transform the graph of a function by adding, subtracting, or multiplying either the “inside” or “outside” of the function by a constant (i.e., by a real number).

In this unit, we will focus on moving the graph of a function horizontally to the left or right (i.e., horizontal shift) and vertically up or down (i.e., vertical shift). These transformations are caused by adding or subtracting constants to the “outside” of a function.

**Vertical Shifts:**

**EXAMPLE:** The graph of \( y = m(x) \) is given below.

Sketch a graph of \( y = n(x) \) if \( n(x) = m(x) + 3 \).

(Check answer: The graph of \( y = m(x) \) shifted up by 3 units.)
**EXAMPLE:** The graph of $y = m(x)$ is given in the previous example. Sketch a graph of $y = q(x)$ if $q(x) = m(x) - 3$.

**SOLUTION:**

Let’s start with a few key points on the graph of $y = m(x)$ and see where they end up on the graph of $y = q(x)$. Let’s use the points $(-4, -1)$, $(2, -1)$, $(1, 2)$, $(2, 0)$, $(3, 3)$ and $(4, 0)$. These are “corner points” on the graph. If we can determine where the corner points end up, we can connect them to see where the rest of the points go.

Let’s start by evaluating $q(-4)$.

$$q(-4) = m(-4) - 3$$

$$= -1 - 3$$

$$= -4$$

Now let’s make a table of values showing where these key ("corner") points end up on the graph of $y = q(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q(x)$</th>
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<tbody>
<tr>
<td>-4</td>
<td>-4</td>
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<tr>
<td>-2</td>
<td>-4</td>
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<tr>
<td>1</td>
<td>-1</td>
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<tr>
<td>2</td>
<td>-3</td>
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<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
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</tbody>
</table>

Finally, let’s plot the points we just found.

Since the points we’ve just plotted represent where the corner points of the original function, we can assume that these points are the corner points of the new function. To obtain the graph of $y = m(x)$ let’s connect these corner points:
The graphs of $q(x) = m(x) - 3$ (purple) and $y = m(x)$ (blue).

**GENERALIZATION:**

If $f$ is a real-valued function and $k \in \mathbb{R}$, then compared with the graph of $y = f(x)$, the graph of $y = f(x) + k$ is shifted vertically $k$ units. (If $k$ is positive, shift up; if $k$ is negative, shift down.)

**EXAMPLE:** Suppose that $d(x)$ represents the height (in inches) of a boy named Dan when he is $x$-months old. If the height of Dan's friend Stan is given by the function $s(x) = d(x) - 3$, what can you say about how the heights of Dan and Stan differ?

**SOLUTION:**

Stan is 3 inches shorter than Dan at any given age because, when Dan is $x$-months old, he is $d(x)$ inches tall, but when Stan is $x$-months old, he is $d(x) - 3$ inches tall.

**EXAMPLE:** Explain how the graph of $g(x) = 3x - 5$ can be transformed into the graph of $h(x) = 3x + 5$.

**SOLUTION:**

In order to answer the question, we need to write $h(x)$ in terms of $g(x)$:

$$h(x) = 3x + 5$$
$$= 3x - 5 + 10$$
$$= g(x) + 10$$

Thus, if we shift the graph of $y = g(x)$ up 10 units we will obtain the graph of $y = h(x)$. 
**Horizontal Shifts:**

**EXAMPLE:** The graph of \( y = m(x) \) is given below. Sketch a graph of \( y = r(x) \) if \( r(x) = m(x - 3) \).

![The graph of \( y = m(x) \)](image_url)

Let's start with a few key points on the graph of \( y = m(x) \) and see where they end up on the graph of \( y = r(x) \). Let's use the points \((-4, -1), (-2, -1), (1, 2), (2, 0), (3, 3)\) and \((4, 0)\).

Let's start by evaluating \( r(-4) \).

\[
r(-4) = m(-4 - 3) = m(-7)
\]

Now we are stuck since \(-7\) is not in the domain of \( m(x) \). What is the first value we can find? We can find \( r(-1) \) because it is the same as \( m(-4) \).

Let's evaluate \( r(-1) \).

\[
r(-1) = m(-1 - 3) = m(-4)
\]

\[
= -1
\]

Let's try evaluating \( r(1) \) next.

\[
r(1) = m(1 - 3) = m(-2)
\]

\[
= -1
\]
Now let’s make a table of values of key points on the graph of $y = r(x)$.

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<tr>
<th>$x$</th>
<th>$r(x)$</th>
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<tbody>
<tr>
<td>-1</td>
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<td>1</td>
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<td>4</td>
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<td>5</td>
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<tr>
<td>6</td>
<td>3</td>
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<tr>
<td>7</td>
<td>0</td>
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</table>

Finally, let’s plot the points we just found.

Now connect the points we’ve plotted to obtain a graph of $y = m(x)$.

The graphs of $r(x) = m(x - 3)$ (purple) and $y = m(x)$ (blue).
EXAMPLE: The graph of \( y = m(x) \) is given below. Sketch a graph of \( y = s(x) \) if \( s(x) = m(x + 3) \).

Since selecting all of the key points in advance didn't work last time, let's just start by evaluating \( s(-4) \) and then decide where to go from there.

\[
s(-4) = m(-4 + 3) \\
= m(-1) \\
= 0
\]

What is the smallest input value we can use? Since \( s(-7) \) is the same as \( m(-4) \) and \(-4\) is the smallest input value on the graph of \( y = m(x) \), \(-4\) is the smallest input value we can use.

\[
s(-7) = m(-7 + 3) \\
= m(-4) \\
= -1
\]

Now let's make a table of values of key ("corner") points on the graph of \( y = s(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( s(x) )</th>
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<tbody>
<tr>
<td>-7</td>
<td>-1</td>
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<tr>
<td>-5</td>
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<td>0</td>
<td>3</td>
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<tr>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Finally, let's plot the points we just found.
Now we can connect the points we've plotted and obtain the graph of $y = s(x)$.

The graphs of $s(x) = m(x + 3)$ (purple) and $y = m(x)$ (blue).

**GENERALIZATION:**

If $f$ is a real-valued function and $h \in \mathbb{R}$, then compared with the graph of $y = f(x)$, the graph of $y = f(x - h)$ is shifted horizontally $h$ units. If $h$ is positive, shift to the right; if $h$ is negative, shift to the left.

**BE CAREFUL:** Notice that there is a subtraction symbol in the rule $g(x) = f(x - h)$.
So if $g(x) = f(x + 3)$, then $h = -3$, and we must shift left 3 units.
EXAMPLE: Suppose that \( d(x) \) represents the height (in feet) of a boy named Dan when he is \( x \) months old. If the height of Dan’s friend Fran is given by the function \( f(x) = d(x - 3) \), what can you say about how the heights of Dan and Fran differ?

SOLUTION:

To answer the question, let's first consider a few particular examples: What if Fran is 10 months old? Since the definition of the function \( f \) tells us that

\[
f(10) = d(10 - 3) = d(7).
\]

Thus, when Fran is 10 months old she is the same height as Dan when he was 7 months old. Similarly, if Fran is 3 months old, she is the same height that Dan was when he was born since \( f(3) = d(0) \). We can use these particular examples to conclude that Fran as tall as Dan was 3 months earlier. Note that the function \( f \) only makes sense when \( x \geq 3 \), i.e., the domain of \( f \) is \( \{ x \mid x \in \mathbb{R} \text{ and } x \geq 3 \} \).

EXAMPLE: Explain how the graph of \( g(x) = x^2 \) can be transformed into the graph of \( z(x) = (x + 5)^2 + 1 \).

SOLUTION:

We need to be able to write \( z(x) \) in terms of \( g(x) \) in order to be sure that we can transform the graph of \( y = g(x) \) into the graph of \( y = z(x) \).

\[
z(x) = (x + 5)^2 + 1 = g(x + 5) + 1
\]

Now we can list the necessary sequence of transformations:

We obtain the graph of \( y = z(x) \) by:

1\textsuperscript{st}: shifting the graph of \( y = g(x) \) left 5 units.

2\textsuperscript{nd}: shifting the graph up 1 unit.