Section I: Functions and Their Graphs

Unit 3: The Algebra of Functions

We can use the four basic arithmetic operations (addition, subtraction, multiplication, and division) to create new functions from old ones.

**DEFINITION:** If \( f \) and \( g \) are functions and \( x \) represents a value in both of their domains, then we can define the following four functions:

- **Sum-of-Functions:** \((f + g)(x) = f(x) + g(x)\)
- **Difference-of-Functions:** \((f - g)(x) = f(x) - g(x)\)
- **Product-of-Functions:** \((f \cdot g)(x) = f(x) \cdot g(x)\)
- **Quotient-of-Functions:** \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0\)

Here it is important to note that \( g(x) \neq 0 \) since this would cause division by zero, and division by zero is undefined.

**EXAMPLE:** Suppose that the function \( s = f(t) \) represents the total number of female students enrolled at PCC \( t \) years after 1990 and that \( s = m(t) \) represents the total number of male students enrolled at PCC \( t \) years after 2000. Write an expression that represents the total number of students enrolled at PCC \( t \) years after 2000.

**SOLUTION:** \( s = (f + m)(t) \) represents the total number of students enrolled at PCC \( t \) years after 2000.
**EXAMPLE:** Let \( h(x) = 5 - 7x \) and \( k(x) = \frac{10}{1 + x^2} \).

a. Find and simplify the rule for \((h + k)(x)\).

b. Find and simplify the rule for \((h \cdot k)(x)\).

c. Evaluate \((h + k)(2)\).

d. Evaluate \(\left(\frac{h}{k}\right)(1)\).

e. Evaluate \((k - h)(-3)\).

**SOLUTIONS:**

a. \((h + k)(x) = h(x) + k(x)\)

\[
= (5 - 7x) + \left(\frac{10}{1 + x^2}\right)
\]

\[
= (5 - 7x) \cdot \frac{1 + x^2}{1 + x^2} + \frac{10}{1 + x^2}
\]

\[
= \frac{(5 - 7x)(1 + x^2)}{1 + x^2} + \frac{10}{1 + x^2}
\]

\[
= \frac{(5 - 7x)(1 + x^2) + 10}{1 + x^2}
\]

\[
= -7x^3 + 5x^2 - 7x + 15
\]

\[
= \frac{1 + x^2}{1 + x^2}
\]

b. \((h \cdot k)(x) = (5 - 7x) \cdot \left(\frac{10}{1 + x^2}\right)\)

\[
= \frac{10(5 - 7x)}{1 + x^2}
\]

\[
= \frac{50 - 70x}{1 + x^2}
\]
c. We can do this one two ways. First, we’ll use the formula we found in a. Since

\[
(h+k)(x) = \frac{-7x^3 + 5x^2 - 7x + 15}{1 + x^2}
\]

we see that

\[
(h+k)(2) = \frac{-7(2)^3 + 5(2)^2 - 7(2) + 15}{1 + (2)^2}
\]

\[
= \frac{-56 + 20 - 14 + 15}{1 + 4}
\]

\[
= \frac{-35}{5}
\]

\[
= -7
\]

Alternatively, we can calculate \( h(2) \) and \( k(2) \) separately, and then add the results:

\[
(h + k)(2) = h(2) + k(2)
\]

\[
= (5 - 7(2)) + \left( \frac{10}{1 + (2)^2} \right)
\]

\[
= -9 + \frac{10}{5}
\]

\[
= -9 + 2
\]

\[
= -7
\]

d. As with c, we can do this two ways. Since we haven't yet found a formula for \( \left( \frac{h}{k} \right)(x) \), we’ll just calculate \( h(1) \) and \( k(1) \), and then divide the results.

\[
\left( \frac{h}{k} \right)(1) = \frac{h(1)}{k(1)}
\]

\[
= \frac{5 - 7(1)}{\left( \frac{10}{1 + (1)^2} \right)}
\]

\[
= \frac{-2}{5}
\]

\[
= -\frac{2}{5}
\]


\[
\begin{align*}
e. \quad (k - h)(-3) &= k(-3) - h(-3) \\
&= \left( \frac{10}{1 + (-3)^2} \right) - (5 - 7(-3)) \\
&= \left( \frac{10}{1 + 9} \right) - (5 + 21) \\
&= 1 - (26) \\
&= -25
\end{align*}
\]

**EXAMPLE:** Given the graphs of \( y = f(x) \) and \( y = g(x) \) in Figures 1 and 2, respectively, graph \( y = (f + g)(x) \) in Figure 3.

![Figure 1: Graph of \( y = f(x) \).](image1)

![Figure 2: Graph of \( y = g(x) \).](image2)

**SOLUTION:**

To graph \( y = (f + g)(x) \), choose an input value and add the corresponding output values. For instance, \( f(4) = 4 \) and \( g(4) = 1 \), so \( (f + g)(4) = 5 \), while \( (f + g)(-6) = -2 \) since \( f(-6) = -1 \) and \( g(-6) = -1 \).

![Figure 3: Graph of \( y = (f + g)(x) \).](image3)
Try this one yourself and check your answer.

If \( f(x) = 2x - 1 \) and \( g(x) = -x + 3 \), find

\[
\begin{align*}
\text{a. } & (f + g)(x) \\
\text{b. } & (f - g)(x) \\
\text{c. } & (f \cdot g)(x) \\
\text{d. } & \left(\frac{f}{g}\right)(x)
\end{align*}
\]

SOLUTIONS:

\[
\begin{align*}
\text{a. } & \text{ Click here for solution} \\
\text{b. } & \text{ Click here for solution} \\
\text{c. } & \text{ Click here for solution} \\
\text{d. } & \left(\frac{f}{g}\right)(x) = \frac{2x - 1}{-x + 3} \text{ for } x \neq 3 \quad \text{(it is important to note that } x \neq 3 \text{ since this would cause division by zero, and division by zero is undefined)}
\end{align*}
\]

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Try this one yourself and check your answer.

Fill in the missing parts of the table below.

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<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>((f + g)(x))</th>
<th>((f - g)(x))</th>
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\[\text{CLICK HERE FOR SOLUTION}\]