A function is a special type of binary relation. So before we discuss what a function is, we need to define binary relation.

**DEFINITION:** A binary relation is a rule that establishes a relationship between two sets. (Note that this is an informal definition.)

The two sets involved in a binary relation play different roles; these roles are determined by the rule of the relation (see the first example). The two roles that the sets play are the “first set” or the “set of inputs” and the “second set” or the “set of outputs.” The set of inputs is called the domain; the set of outputs is called the range.

**EXAMPLE:**

- **a.** If the rule of the relation is “Associate each college student with the college that the student currently attends” then the domain is the set of college students and the range is the set of colleges.

- **b.** If the rule of the relation is “Associate each college with each student who currently attends that college,” then the domain is the set of colleges and the range is the set of college students.

**EXAMPLE:**

- **a.** If the rule of the relation is “Associate each person with his/her social security number” then the domain is the set of people and the range is the set of social security numbers.

- **b.** If the rule of the relation is “Associate each social security number with the person who has been assigned to that number,” then the domain is the set of social security numbers and the range is the set of people.

**EXAMPLE:** Consider a binary relation defined on the following two sets:

- **Input Set (domain):** \{Tom, Ann, Ken, Sam\}
- **Output Set (range):** \{Cat, Dog, Rat\}

The rule that relates these sets is: “Each person (from the input set) is related to an animal (from the output set) that he or she has as a pet.” The arrow diagram below defines this relation.
The arrow diagram tells us, for example that “Sam is related to Dog” (since there is an arrow from Sam to Dog), and this means that Sam has a dog. We also see that “Sam is related to Rat,” so Sam also has a Rat.

Another way to convey the information contained in a relation is to use ordered pairs. To represent the fact that “Sam is related to Dog” we use an ordered pair: \((Sam, Dog)\). The first term in each ordered pair is an element of the set of inputs and the second term in each ordered pair is an element of the set of outputs. The entire relation represented by the arrow diagram above can be represented by the set of ordered pairs below:

\[\{(Tom, Dog), (Ann, Cat), (Ann, Dog), (Ann, Rat), (Ken, Cat), (Sam, Dog), (Sam, Rat)\}\]

**DEFINITION:** A function is a binary relation in which each element in the domain corresponds to exactly one element in the range.

**EXAMPLE:** The arrow diagram at the right represents a binary relation on the sets \{Sue, Pablo, Kim, Ron\} and \{Boy, Girl\}. The rule that defines this relation is: “Associate each person with his/her gender.” This relation is a function since each person has exactly one gender. When viewing the arrow diagram, we can tell that this is a function since there is exactly one arrow coming out of each element in the domain. (Notice that, in the example above, more than one arrow comes from some elements in the domain, so the diagram does not represent a function.) We can represent this function by the following set of ordered pairs:

\[\{(Sue, Girl), (Pablo, Boy), (Kim, Girl), (Ron, Boy)\}\]
EXAMPLE: Determine whether each of the following relations describes a function with domain \{1, 2, 3, 4\} and range \{5, 6, 7, 8\}.

\[\begin{align*}
a. \quad & \{(1, 5), (2, 6), (3, 7), (4, 8)\} \\ b. \quad & \{(1, 7), (1, 8), (2, 5), (3, 6)\} \\ c. \quad & \{(1, 5), (2, 5), (3, 5), (4, 5)\}
\end{align*}\]

SOLUTIONS:

\[\begin{align*}
a. \quad & \{(1, 5), (2, 6), (3, 7), (4, 8)\} \text{ represents a function because each element in the domain corresponds to } exactly \text{ one element in the range.} \\ b. \quad & \{(1, 7), (1, 8), (2, 5), (3, 6)\} \text{ does not represent a function because the number } 1 \text{ corresponds to } two \text{ different elements of the range.} \\ c. \quad & \{(1, 5), (2, 5), (3, 5), (4, 5)\} \text{ is a function because each element in the domain corresponds to } exactly \text{ one element in the range. (Note that it is okay to repeat elements in the range.)}
\end{align*}\]

EXAMPLE: The table below defines the function \(f\), where the inputs come from \(A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}\) and the outputs come from \(B = \{-5, -3, 0, 1, 2, 3, 4, 9, 13\}\). Thus, the set \(A\) is the domain of \(f\) while the set \(B\) is the range of \(f\), which we can represent with the notation \(f : A \rightarrow B\).

<table>
<thead>
<tr>
<th>DOMAIN OF (f)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RANGE OF (f)</td>
<td>(4)</td>
<td>(-3)</td>
<td>(-5)</td>
<td>(4)</td>
<td>(1)</td>
<td>(3)</td>
<td>(0)</td>
<td>(9)</td>
<td>(-3)</td>
<td>(2)</td>
<td>(13)</td>
</tr>
</tbody>
</table>

If someone were to ask you, “Where does \(f\) send \(-2\)?” you would (hopefully) say, “\(f\) sends \(-2\) to \(4\)” (since there is a column in the table showing that the element \(-2\) in the domain is related to \(4\) in the range). Function notation offers another way to communicate this information:

\[f(-2) = 4 \text{ means } \text{“}f\text{ sends } -2 \text{ to } 4.\]

Use the table above to evaluate the following:

\[\begin{align*}
a. \quad & \text{Evaluate } f(-4). \\ b. \quad & \text{Evaluate } f(1). \\ c. \quad & \text{Evaluate } f(9).
\end{align*}\]
Sometimes we don’t know what the input is. In such a case, we can call the input \( x \) (or any other variable), where \( x \) represents a generic element of the domain; then \( f(x) \) represents the element in the range that \( x \) is associated with under the function \( f \). In this example, the domain of \( f \) is the set \( A \) and the range is the set \( B \). Thus, \( x \in A \) and \( f(x) \in B \). With this new notation, we can rename the rows in our table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(4)</td>
<td>(−3)</td>
<td>(−5)</td>
<td>(4)</td>
<td>(1)</td>
<td>(3)</td>
<td>(0)</td>
<td>(9)</td>
<td>(−3)</td>
<td>(2)</td>
<td>(13)</td>
</tr>
</tbody>
</table>

Solve the following for \( x \).

d. Solve \( f(x) = 1 \).

e. Solve \( f(x) = −3 \).

f. Solve \( f(x) = 5 \).

Equations as Functions

Equations in two variables often represent functions since one variable can represent an element from the domain and the other variable can represent an element from the domain. For example, the equation \( y = x + 3 \) defines a function, where the domain value is represented by \( x \) and the range value is represented by \( y \). It is often helpful to write equations like \( y = x + 3 \) in function notation, so that the value in the range that is associated with the domain value “\( x \)” is represented as “\( f(x) \)”: \[
y = x + 3 \Rightarrow f(x) = x + 3
\]

which is read “\( f \) of \( x \) equals \( x \) plus 3”.

Similarly, the equation \( u = 3t − 5 \) can be written in function notation as \( g(t) = 3t − 5 \). This is read: “\( g \) of \( t \) equals 3\( t \) minus 5.”

Caution: The notation \( f(x) \) does not mean multiplication of \( f \) by \( x \).

Function notation involving \( f(x) \) is more compact than notation involving "\( y = \). For example:

With function notation: If \( f(x) = x + 3 \), then \( f(5) = 8 \).

Without function notation: If \( y = x + 3 \), then the value of \( y \) is 8 when \( x \) is 5.

The independent variable refers to the variable representing possible values in the domain, and the dependent variable refers to the variable representing possible values in the range. Thus in our usual ordered pair notation \((x, y)\), \( x \) is the independent variable and \( y \) is the dependent variable. It is customary to plot the independent variable on the horizontal axis and the dependent variable on the vertical axis.
EXAMPLE: Find the indicated function value.

a. \( f(7) \) if \( f(x) = 3x - 5 \)

b. \( g(-3) \) if \( g(x) = x^2 - x + 4 \)

c. \( h(-1) \) if \( h(t) = -t + 5 \)

d. \( k(0) \) if \( k(m) = m^{50} + 13m^{36} - 62m^4 + 6 \)

SOLUTIONS:

a. \( f(7) = 3(7) - 5 \)
   \[ = 21 - 5 \]
   \[ = 16 \]

b. \( g(-3) = (-3)^2 - (-3) + 4 \)
   \[ = 9 + 3 + 4 \]
   \[ = 16 \]

c. \( h(-1) = -(-1) + 5 \)
   \[ = 1 + 5 \]
   \[ = 6 \]

d. \( k(0) = (0)^{50} + 13(0)^{36} - 62(0)^4 + 6 \)
   \[ = 6 \]

EXAMPLE: Suppose that \( f(x) = x^2 - x \). Find

a. \( f(2a) \).

b. \( f(a + b) \).

c. \( f(a) + b \).
SOLUTIONS:

a. \( f(2a) = (2a)^2 - 2a \)
   \[ = 4a^2 - 2a \]

b. \( f(a + b) = (a + b)^2 - (a + b) \)
   \[ = (a + b)(a + b) - a - b \]
   \[ = a^2 + 2ab + b^2 - a - b \]

c. \( f(a) + b = (a)^2 - a + b \)
   \[ = a^2 - a + b \]

EXAMPLE: Suppose that \( g(x) = 3x \). Find

a. \( g(x + 2) \).

b. \( g(x) + 2 \).

c. \( g(2x) \).

d. \( g(2x - 2) \).

SOLUTIONS:

a. \( g(x + 2) = 3(x + 2) \)
   \[ = 3x + 6 \]

b. \( g(x) + 2 = 3x + 2 \)

c. \( g(2x) = 3(2x) \)
   \[ = 6x \]

d. \( g(2x - 2) = 3(2x - 2) \)
   \[ = 6x - 6 \]
Graphs on the two dimensional coordinate plane can represent functions, where (traditionally) the values on the horizontal axis represent the domain and values on the vertical axis represent the range.

**EXAMPLE:** The function $f$ is graphed in Figure 1 below. (Note that there is an arrow on the left-hand side of the graph since the function doesn’t end when the graph ends. If the graph ends, it will be denoted with a filled-in circle, as there is on the right-hand side.) Determine the following:

a. $f(1)$

b. the domain of $f$

c. any $x$-value(s) for which $f(x) = 2$

d. any $x$-value(s) for which $f(x) = -1$

e. the range of $f$

![Figure 1: The graph of $y = f(x)$.](image)

**SOLUTIONS:**

a. $f(1) = -3$ which can be determined because the point $(1, -3)$ is on the graph.

b. The domain of $f$ is $(-\infty, 3]$ since the graph suggests that there is a point above or below every $x$-value less than or equal to 3; every $x$-value less than or equal to 3 is “used” by the function so it's in the domain.

c. $f(x) = 2$ when $x \approx -2.2$ because $f(-2.2) \approx 2$. (Notice that we use “≈” to denote the fact that we are approximating the answer.)

d. $f(x) = -1$ when $x = -1$ or $x = 3$.

e. The range of $f$ is $[-3, \infty)$ since the graph suggests that there is a point to the left or right of every $y$-value greater than or equal to $-3$; every $y$-value greater than or equal to $-3$ is “used” by the function so it's in the range.
EXAMPLE: The function $p$ is graphed in Figure 2 below. Find the following:

a. $p(1)$.

b. the domain of $p$.

c. all $x$-values for which $p(x) = 2$.

d. the range of $p$.

Figure 2: The graph of $y = p(x)$.

SOLUTIONS:

a. $p(1) = 1$ because the point $(1, 1)$ is on the graph. Note that there is a hole at the point $(1, -3)$, so the point $(1, -3)$ is NOT on the graph.

b. The domain of $p$ is $[-5, 3]$ (in interval notation) or $\{x \in \mathbb{R} \text{ and } -5 \leq x \leq 3\}$ (in set-builder notation).

c. The $x$-values for which $p(x) = 2$ are $x = -4$ and $x \approx -1.5$.

d. The range of $p$ in interval notation is $(-3, -1] \cup [0, 4]$.
**EXAMPLE:** The function \( y = g(x) \) is defined in Figure 3, the function \( y = f(x) \) is defined in Table 1, and \( h(x) = x^2 + 1 \).

![Figure 3: The graph of \( y = g(x) \).](image)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(3)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(5)</td>
</tr>
<tr>
<td>(0)</td>
<td>(-2)</td>
</tr>
<tr>
<td>(1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(2)</td>
<td>(7)</td>
</tr>
</tbody>
</table>

Table 1

Find each of the following:

a. \( f(1) \)  

b. \( g(-1) \)  

c. \( h(-3) \)  

d. \( f(g(4)) \)  

e. \( h(2x + 3) \)  

f. \( x \) if \( f(x) = 7 \)  

g. \( x \) if \( h(x) = 5 \)  

h. \( x \) if \( g(x) = 0 \)

**SOLUTIONS:**

a. \( f(1) = -1 \) (using the table)

b. \( g(-1) = 4 \) (using the graph)  

Note that \( g(-1) \neq 3 \) since the point \((-1, 3)\) isn’t on the graph. A solid circle represents a point on the graph, but the open circle at \((-1, 3)\) represents a hole in the graph.

c. \( h(-3) = (-3)^2 + 1 \)  

\[= 9 + 1\]  

\[= 10\]

d. \( f(g(4)) = f(-1) \) (first calculate \( g(4) \) using the graph)  

Then calculate \( f(-1) \) from the table.

e. \( h(2x + 3) = (2x + 3)^2 + 1 \) (the input is the entire expression \( 2x + 3 \))  

\[= (2x + 3)(2x + 3) + 1\]  

\[= 4x^2 + 6x + 6x + 9 + 1\]  

\[= 4x^2 + 12x + 10\]

f. Since \( f(2) = 7 \), if \( f(x) = 7 \), then \( x = 2 \). Look for an output value of 7 in the table.
g. To solve this equation we need to find an output value of 0 on the graph of \( y = g(x) \). (So we are looking for the \( x \)-intercepts on the graph of \( y = g(x) \).) Since \( g(-3) = 0 \) and \( g(3) = 0 \), the solution is \( "x = -3 \text{ or } x = 3". \) To communicate this using set notation, we can write \( "x \in \{-3, 3\}" \) or \( "The solution set for \( g(x) = 0 \) is \( \{-3, 3\}\)." \)

h. \[ h(x) = 5 \]
\[ \Rightarrow x^2 + 1 = 5 \]
\[ \Rightarrow x^2 = 4 \]
\[ \Rightarrow x = 2 \text{ or } x = -2 \]

Recall that the definition of a function requires that each input has only one corresponding output. Since the inputs are \( x \)-values and the outputs are \( y \)-values, a function can have only one \( y \)-value for each \( x \)-value. If we translate this information to the context of the graph of a function, we quickly realize that (since each \( x \)-value is represented by a vertical line) the graph of a function can have at most one point on each vertical line. (If a vertical line \( x = a \) passes through a graph more than one time, then the \( x \)-value \( a \) has more than one corresponding output, so the graph cannot represent a function.) This leads us to the **Vertical Line Test**:

**The Vertical-Line Test**

If you graph a relation on the \( xy \)-plane and every vertical line intersects the graph at most once, then \( y \) is a function of \( x \). (If it is possible to draw a vertical line that intersects the graph more than once, then \( y \) is not a function of \( x \).)

**EXAMPLE:** In which of the following graphs is \( y \) a function of \( x \)?

**SOLUTION:** In Figures 4 and 6, it is possible to draw vertical lines that intersect the graphs more than once (e.g., the \( y \)-axis is a vertical line that intersects these graphs more than once), so \( y \) is not a function of \( x \) in these graphs. In Figure 5, the graph passes the vertical line test since no vertical line intersects the graph more than once, so \( y \) a function of \( x \) in this graph.
In the table below, some important terms concerning the behavior of functions and their graphs are described. (For the most part, these are “casual descriptions” not “formal definitions”.)

The **zeros** of a function $f$ are the $x$-values that correspond to the output value $0$, i.e., if the input value $a$ is a zero of $f$ then $f(a) = 0$, i.e., if $a$ is a zero of $f$ then $(a, 0)$ is an $x$-intercept of $f$.

A function has a **local maximum** at the input value $a$ if the output value at $a$ is larger than any other output value in an interval near $a$. A function has a **local minimum** at the input value $a$ if the output value at $a$ is smaller than any other output value in an interval near $a$.

A function $f$ is **increasing** if it “goes uphill” as you move along the curve from left to right. A function $f$ is **decreasing** if it "goes downhill" as you move along the curve from left to right.

A function is **concave up** if it is “curved in a U-shape”. (An upward opening parabola is everywhere concave up.) A function is **concave down** if it is “curved in an upside down U-shape”. (A downward opening parabola is everywhere concave down.)

**EXAMPLE:**

a. What are the zeros of $y = m(x)$?

b. On what interval(s) is $y = m(x)$ increasing?

c. On what interval(s) is $y = m(x)$ decreasing?

d. On what interval(s) is $y = m(x)$ concave up?

e. On what interval(s) is $y = m(x)$ concave down?

f. Does $y = m(x)$ have a local minimum? If so, where does it occur?

g. Does $y = m(x)$ have a local maximum? If so, where does it occur?

![Figure 7: The graph of $y = m(x)$.](image-url)
SOLUTIONS:

a. The zeros of \( y = m(x) \) are \(-3, -1, \text{ and } 1\).

b. \( y = m(x) \) is increasing in the interval \((-2, 0)\).

c. \( y = m(x) \) decreasing on the intervals \((-\infty, -2) \text{ and } (0, \infty)\).

d. \( y = m(x) \) is concave up on the interval \((-1, \infty)\).

e. \( y = m(x) \) is concave down on the interval \((-\infty, -1)\).

f. \( y = m(x) \) has a local minimum at \( x = -2 \).

g. \( y = m(x) \) has a local maximum at \( x = 0 \).

---

**Back-pocket Functions**

Throughout Section I of the online lecture notes, we will study how we can create new functions from old ones. But the only way we can do this is if we are already familiar with some “old functions”. The following eight functions are some of the most important since they are often the building-blocks we use to form more complicated functions. I call these functions “back-pocket functions” since you should always carry with you their graphs in your figurative back-pocket, i.e., you should know these graphs! (See pages 82–83 in our textbook.)

a. \( f(x) = x \)

\[ \begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & \mathbb{R}
\end{align*} \]

b. \( f(x) = x^2 \)

\[ \begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & [0, \infty)
\end{align*} \]
c. \( f(x) = x^3 \)

\[
\begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & \mathbb{R}
\end{align*}
\]

d. \( f(x) = |x| \)

\[
\begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & [0, \infty)
\end{align*}
\]

e. \( f(x) = \sqrt{x} \)

\[
\begin{align*}
\text{Domain: } & [0, \infty) \\
\text{Range: } & [0, \infty)
\end{align*}
\]

f. \( f(x) = \sqrt[3]{x} \)

\[
\begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & \mathbb{R}
\end{align*}
\]

g. \( f(x) = \frac{1}{x} \)

\[
\begin{align*}
\text{Domain: } & (-\infty, 0) \cup (0, \infty) \\
\text{Range: } & (-\infty, 0) \cup (0, \infty)
\end{align*}
\]

h. \( f(x) = \frac{1}{x^2} \)

\[
\begin{align*}
\text{Domain: } & (-\infty, 0) \cup (0, \infty) \\
\text{Range: } & (0, \infty)
\end{align*}
\]