Section II: Exponential and Logarithmic Functions

Unit 5: Logarithmic Functions and Equations

**EXAMPLE:** Consider the exponential function \( k(x) = 2^x \) which is graphed in Figure 1.

![Graph of \( y = k(x) \)](image)

**Figure 1:** The graph of \( y = k(x) \).

Does \( k \) have an inverse?

The graph of \( k(x) = 2^x \) passes the horizontal line test (no horizontal line crosses the graph of \( y = k(x) \) more than once), which implies that each output (or \( x \)-value) comes from exactly one input (or \( x \)-value). Thus, \( k \) is one-to-one, so \( k \) has an inverse function.

**Sketch a graph of \( y = k^{-1}(x) \).**

As we learned in Section I: Unit 6, the graph of the inverse of a function is the reflection of the function about the line \( y = x \). So, to graph \( y = k^{-1}(x) \), we need to reflect the graph of \( y = k(x) \) about the line \( y = x \).

![Graphs of \( y = k(x) \), \( y = k^{-1}(x) \), and \( y = x \)](image)

**Figure 2:** The graphs of \( y = k(x) \), \( y = k^{-1}(x) \), and \( y = x \).
We call the inverses of exponential functions **logarithmic functions**. So the inverse of \( k(x) = 2^x \) is called the **logarithmic function of base 2** and is written \( k^{-1}(x) = \log_2(x) \). Notice that the **domain of** \( k^{-1}(x) = \log_2(x) \) is \((0, \infty)\).

Since all exponential functions are one-to-one (i.e., all exponential functions have graphs that are similar to \( k(x) = 2^x \) and pass the horizontal line test), we can conclude that all exponential functions have inverse functions. The inverse of an exponential function is a **logarithmic function**. The domain of a logarithmic function is \((0, \infty)\).

The inverse of the exponential function \( f(x) = a^x \) (where \( a > 0 \)) is the function \( f^{-1}(x) = \log_a(x) \), the **logarithm of base** \( a \).

**EXAMPLE:** If \( h(x) = 7^x \), then the inverse of \( h \) is the function \( h^{-1}(x) = \log_7(x) \).

**EXAMPLE:** Consider the function \( g(x) = 10^x \). Since \( g \) is an exponential function, its inverse is a logarithmic function. Both \( g \) and \( g^{-1} \) are graphed in Figure 3.

![Figure 3: The graphs of \( y = g(x) \), \( y = g^{-1}(x) \), and \( y = x \).](image)

The inverse of \( g(x) = 10^x \) is \( g^{-1}(x) = \log_{10}(x) \), the **logarithm of base 10**. This is an important and often-used function so it is given a special name, the **common logarithm**, and is usually written as “\( \log(x) \)” instead of “\( \log_{10}(x) \)”.

---

### 2. Logarithmic Functions

- **Logarithmic Functions**
  - **Logarithmic function of base 2**: \( k^{-1}(x) = \log_2(x) \)
  - **Domain**: \((0, \infty)\)

- **Logarithm of base** \( a \): \( f^{-1}(x) = \log_a(x) \)

- **Domain of a logarithmic function**: \((0, \infty)\)

---

### 3. Examples

- **Example 1**: \( h(x) = 7^x \), inverse: \( h^{-1}(x) = \log_7(x) \)

- **Example 2**: \( g(x) = 10^x \), inverse: \( g^{-1}(x) = \log_{10}(x) \), **common logarithm**

---

### 4. Graphs

- **Figure 3**: Graphs of \( y = g(x) \), \( y = g^{-1}(x) \), and \( y = x \)
Now let's investigate the relationship between the inputs and outputs of a logarithmic function.

**EXAMPLE:** Consider \( g(x) = 10^x \) and \( g^{-1}(x) = \log(x) \). See Tables 1 and 2 below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( g^{-1}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>–2</td>
<td>( \frac{1}{100} )</td>
<td>( \frac{1}{100} )</td>
<td>–2</td>
</tr>
<tr>
<td>–1</td>
<td>( \frac{1}{10} )</td>
<td>( \frac{1}{10} )</td>
<td>–1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>1000</td>
<td>3</td>
</tr>
</tbody>
</table>

Recall that the inputs become the outputs and the outputs become the inputs when we create the inverse of a function. Since the input into exponential functions (like \( g(x) = 10^x \)) are exponents, the outputs of logarithmic functions (like \( g^{-1}(x) = \log(x) \)) are exponents. Thus, it is useful to keep the following sentence in mind while working with logarithms.

**“The outputs for logarithms are exponents.”**

**EXAMPLE:** Simplify the following:

a. \( \log_2(8) \)  
b. \( \log_5(125) \)  
c. \( \log_9\left(\frac{1}{3}\right) \)

**SOLUTIONS:**

a. \( \log_2(8) = \log_2\left(2^3\right) = 3 \)

b. \( \log_5(125) = \log_5\left(5^3\right) = 3 \)
c. \( \log_9 \left( \frac{1}{3} \right) = \log_9 \left( 3^{-1} \right) \)
   
   \[ = \log_9 \left( \left(9^{1/2} \right)^{-1} \right) \]
   
   \[ = \log_9 \left( 9^{-1/2} \right) \]
   
   \[ = -\frac{1}{2} \]

Recall the exponential function \( g(x) = e^x \). Like all exponential functions, \( g(x) = e^x \) is one-to-one, so it has an inverse: \( g^{-1}(x) = \log_e(x) \). Just as \( g(x) = e^x \) is an important function, its inverse is important and, therefore, has a special name: the natural logarithm, and is usually written as “\( \ln(x) \)” instead of “\( \log_e(x) \)”.

**EXAMPLE:** Simplify the following:

a. \( \ln \left( e^4 \right) \)  
b. \( \ln \left( \frac{1}{e} \right) \)  
c. \( \ln(e) \)

**SOLUTIONS:**

a. \( \ln \left( e^4 \right) = 4 \quad \text{(since the natural logarithm has base } e) \)

b. \( \ln \left( \frac{1}{e} \right) = \ln \left( e^{-1} \right) \)
   
   \[ = -1 \]

c. \( \ln(e) = \ln \left( e^1 \right) \)
   
   \[ = 1 \]
Properties of Logarithms

There are a few “properties” that are unique to logarithmic expressions. As you will (hopefully) notice these properties are analogous to the laws of exponents. Let’s start with the Log-of-Products property.

**LOG-OF-PRODUCTS PROPERTY:**

If \( m, n, a \in \mathbb{R}^+ \), then \( \log_a (m \cdot n) = \log_a (m) + \log_a (n) \).

**PROOF:** Let \( x = \log_a (m) \) and \( y = \log_a (n) \) which implies that \( m = a^x \) and \( n = a^y \). Then:

\[
\log_a (m \cdot n) = \log_a (a^x \cdot a^y) \\
= \log_a (a^{x+y}) \\
= x + y \\
= \log_a (m) + \log_a (n) \\
\]

(Using the log-of-products property)

(Using a property of exponents)

(since \( y = a^x \) and \( y = \log_a (x) \) are inverse functions)

\( x = \log_a (m) \) and \( y = \log_a (m) \)

**EXAMPLE:** \( \log(2) + \log(5) \)

**SOLUTION:**

\[
\log(2) + \log(5) = \log(2 \cdot 5) \quad \text{(using the log-of-products property)} \\
= \log(10) \\
= 1
\]
**LOG-OF-POWERS PROPERTY:**

If $m, a \in \mathbb{R}^+$ and $p \in \mathbb{R}$, then $\log_a (m^p) = p \cdot \log_a (m)$.

**PROOF:** Let $x = \log_a (m)$ which implies that $m = a^x$. Then:

\[
\log_a (m^p) = \log_a \left( \left( a^x \right)^p \right) \quad \text{(since } m = a^x \text{)} \\
= \log_a (a^{xp}) \quad \text{(using a property of exponents)} \\
= x \cdot p \quad \text{(since the } y = a^x \text{ and } y = \log_a (x) \text{ are inverse functions)} \\
= p \cdot \log_a (m) \quad \text{(since } x = \log_a (m) \text{)}
\]

**EXAMPLE:** Solve $\log_3 (9^x) = 4$ for $x$.

**SOLUTION:**

\[
\log_3 (9^x) = 4 \\
\Rightarrow x \cdot \log_3 (9) = 4 \quad \text{(using the log-of-powers property)} \\
\Rightarrow x \cdot \log_3 (3^2) = 4 \\
\Rightarrow x \cdot 2 = 4 \\
\Rightarrow x = 2
\]
**LOG-OF-QUOTIENTS PROPERTY:**

If $m, n, a \in \mathbb{R}^+$, then

$$\log_a \left( \frac{m}{n} \right) = \log_a (m) - \log_a (n).$$

**PROOF:** Our proof will utilize the log-of-products and log-of-powers properties.

Let $x = \log_a (m)$ and $y = \log_a (n)$. Then:

$$\log_a \left( \frac{m}{n} \right) = \log_a \left( m \cdot \frac{1}{n} \right)$$

$$= \log_a (m) + \log_a \left( \frac{1}{n} \right) \quad \text{(using the log-of-products property)}$$

$$= \log_a (m) + \log_a (n^{-1}) \quad \text{(using a property of exponents)}$$

$$= \log_a (m) + (-1) \cdot \log_a (n) \quad \text{(using the log-of-powers property)}$$

$$= \log_a (m) - \log_a (n)$$

**EXAMPLE:** $\log_3(12) - \log_3(4)$

**SOLUTION:**

$$\log_3(12) - \log_3(4) = \log_3 \left( \frac{12}{4} \right) \quad \text{(using the log-of-quotients property)}$$

$$= \log_3(3)$$

$$= 1$$
**Solving Logarithmic Equations**

**EXAMPLE:** Solve \( \log_3(x^2 - 8x) = 2 \) for \( x \).

**SOLUTION:**

\[
\log_3(x^2 - 8x) = 2 \\
\Rightarrow x^2 - 8x = 3^2 \\
\Rightarrow x^2 - 8x - 9 = 0 \\
\Rightarrow (x - 9)(x + 1) = 0 \\
\Rightarrow x = 9 \text{ or } x = -1
\]

**CHECK:**

We need to check our work. It is always a good idea to check, but when solving logarithmic equations it is especially important since the properties of logarithms allow for the possibility that we will find TOO MANY solutions. Sometimes we will do ALL of the math correctly, but still get an incorrect solution! **So it is necessary to check your solutions to logarithmic equations in order to rule out any extraneous solutions.**

\[
\begin{align*}
\log_3(x^2 - 8x) &= 2 \\
x = 9 &\Rightarrow \log_3((9)^2 - 8(9)) \overset{?}{=} 2 \\
&\Rightarrow \log_3(81 - 72) \overset{?}{=} 2 \\
&\Rightarrow \log_3(9) \overset{?}{=} 2 \\
&\Rightarrow \log_3(3^2) = 2 \\
\text{So } 9 \text{ is a solution.}
\end{align*}
\]

\[
\begin{align*}
\log_3(x^2 - 8x) &= 2 \\
x = -1 &\Rightarrow \log_3((-1)^2 - 8(-1)) \overset{?}{=} 2 \\
&\Rightarrow \log_3(1 + 8) \overset{?}{=} 2 \\
&\Rightarrow \log_3(9) \overset{?}{=} 2 \\
&\Rightarrow \log_3(3^2) = 2 \\
\text{So } -1 \text{ is a solution.}
\end{align*}
\]

Since both 9 and -1 check, the solution set for the equation \( \log_3(x^2 - 8x) = 2 \) is \( \{9, -1\} \).
EXAMPLE: Solve $\ln(2x + 5) = 0$ for $x$.

SOLUTION:

$$\ln(2x + 5) = 0$$

$\Rightarrow$  
$$2x + 5 = e^0 \quad \text{(first we translate the logarithmic equation into its "exponential" equivalent)}$$

$\Rightarrow$  
$$2x = 1 - 5$$

$\Rightarrow$  
$$x = \frac{-4}{2} = -2$$

CHECK:

$$\ln(2x + 5) = 0$$

$x = -2 \Rightarrow \ln(2(-2) + 5) \neq 0$  
$$\ln(-4 + 5) = 0$$

$\ln(1) = 0$

So $-2$ is a solution.

Therefore, the solution set for the equation $\ln(2x + 5) = 0$ is $\{-2\}$.

EXAMPLE: Solve $\log_2(t) + \log_2(t + 2) = 3$ for $t$.

SOLUTION:

$$\log_2(t) + \log_2(t + 2) = 3$$

$\Rightarrow$  
$$\log_2(t(t + 2)) = 3 \quad \text{(use the log-of-product law to obtain a single logarithmic expression on left side)}$$

$\Rightarrow$  
$$t(t + 2) = 2^3$$

$\Rightarrow$  
$$t^2 + 2t - 8 = 0$$

$\Rightarrow$  
$$(t + 4)(t - 2) = 0$$

$\Rightarrow$  
$$t = -4 \text{ or } t = 2$$
CHECK:

\[ \log_2(t) + \log_2(t + 2) = 3 \]

\[ t = -4 \implies \log_2(-4) + \log_2(-4 + 2) \not= 3 \]

↑

Negative numbers are not in the domains of logarithmic functions. So \(-4\) is not a solution.

\[ \log_2(t) + \log_2(t + 2) = 3 \]

\[ t = 2 \implies \log_2(2) + \log_2(2 + 2) \not= 3 \]

\[ \log_2(2) + \log_2(4) \not= 3 \]

\[ 1 + 2 = 3 \]

So 2 is a solution.

Therefore, the solution set for the equation \( \log_2(t) + \log_2(t + 2) = 3 \) is \( \{2\} \).

EXAMPLE: Solve \( 2 \log_5(m) = \log_5(2m - 1) \) for \( m \).

SOLUTION

This equation has logarithms of the same base on both sides. Since logarithmic functions are one-to-one, once we get isolated logarithmic expressions on both sides of the equation, we can set the "inputs" equal.

\[ 2 \log_5(m) = \log_5(2m - 1) \]

\[ \implies \log_5\left(m^2\right) = \log_5(2m - 1) \] (use the log-of-powers law to obtain an isolated logarithmic expression on the left side)

\[ \implies m^2 = 2m - 1 \] (since the logarithmic functions are one-to-one, the "inputs" must be equal if the outputs are equal)

\[ \implies m^2 - 2m + 1 = 0 \]

\[ \implies (m - 1)(m - 1) = 0 \]

\[ \implies m = 1 \]
CHECK:

\[ 2 \log_5(m) = \log_5(2m - 1) \]

\[ m = 1 \Rightarrow 2 \log_5(1) = \log_5(2(1) - 1) \]

\[ 2 \cdot 0 = \text{?} \log_5(1) \]

\[ 0 = 0 \quad \text{So } 1 \text{ is a solution.} \]

Therefore, the solution set for the equation \( 2 \log_5(m) = \log_5(2m - 1) \) is \( \{1\} \).

EXAMPLE: Solve \( \ln(2) + \ln(3w - 1) = 1 \) for \( w \).

SOLUTION

\[ \ln(2) + \ln(3w - 1) = 1 \]

\[ \Rightarrow \ln(2(3w - 1)) = 1 \quad \text{(use the log-of-powers law to obtain an isolated logarithmic expression on the left side)} \]

\[ \Rightarrow 2(3w - 1) = e^1 \quad \text{(translate the logarithmic equation into its "exponential" equivalent)} \]

\[ \Rightarrow 6w - 2 = e \]

\[ \Rightarrow 6w = e + 2 \]

\[ \Rightarrow w = \frac{e + 2}{6} \]

CHECK:

\[ w = \frac{e + 2}{6} \Rightarrow \ln(2) + \ln\left(3 \left(\frac{e + 2}{6}\right) - 1\right) = 1 \]

\[ \ln(2) + \ln\left(\frac{e + 2}{2} - \frac{3}{2}\right) \quad \text{?} \quad 1 \]

\[ \ln(2) + \ln\left(\frac{e}{2}\right) \quad \text{?} \quad 1 \]

\[ \ln\left(2 \cdot \frac{e}{2}\right) \quad \text{?} \quad 1 \]

\[ \ln(e) = 1 \quad \text{So } \frac{e + 2}{6} \text{ is a solution.} \]

Therefore, the solution set for the equation \( \ln(2) + \ln(3w - 1) = 1 \) is \( \left\{ \frac{e + 2}{6} \right\} \).
Change of Base Formula

Suppose you wanted to estimate \( \log_7(13) \). Most calculators generally only have “buttons” for the natural and common logarithms (i.e., the base 10 and base \( e \) logarithms). Since \( \log_7(13) \) has base 7, we need to change its base in order to estimate it on our calculators. In order to derive the change-of-base formula, let \( x = \log_7(13) \) and solve for \( x \) only using functions that are easy to approximate on a calculator:

\[
\begin{align*}
x &= \log_7(13) \\
\Rightarrow	x^x &= 13 \\
\Rightarrow\ln(x^x) &= \ln(13) \\
\Rightarrow x \cdot \ln(7) &= \ln(13) \\
\Rightarrow x &= \frac{\ln(13)}{\ln(7)}
\end{align*}
\]

So \( \log_7(13) = \frac{\ln(13)}{\ln(7)} \).

Now we can estimate \( \log_7(13) \) since the natural logarithm is programmed into our calculators and we can calculate \( \frac{\ln(13)}{\ln(7)} \).

\[
\log_7(13) = \frac{\ln(13)}{\ln(7)} \approx 1.318
\]

There is no reason why we couldn’t have used a different logarithm, like the common logarithm, when we solved for \( x \) above. Thus, we can obtain the following change of base formula.

**Change-of-Base Formula**

In general, for any \( a, b, x \in \mathbb{R}^+ \), \( \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \).

(This formula is most useful when we take \( b \) to be either 10 or \( e \), since we have the common and natural logarithms in our calculators.)