Section II: Exponential and Logarithmic Functions

Unit 4: Compound Interest and the Number \( e \)

Recall that an exponential function is a function of the form \( f(t) = C \cdot a^t \) where \( a \) is the initial value and \( a = 1 + r \) where \( r \) is the percentage rate of change per units of \( x \).

**EXAMPLE:** Suppose you deposit $1000 in a savings account that gives 5% simple annual interest. (So you get your interest all at once at the end of each year.) Since your money is growing by a constant percentage rate, the function \( m \) that represents the amount of money in your account after \( t \) years is an exponential function. The “initial value” is \( C = 1000 \), and the percent rate of change per unit of \( t \) is 5\%, or 0.05, so \( m(t) = 1000(1 + 0.05)^t \).

**EXAMPLE:** Now suppose that you deposit the same $1000 at the same 5% annual interest, but this time your interest is compounded monthly. This means that every month (i.e., twelve times each year), you get \( \frac{1}{12} \) th of the year’s interest. So you’ll get one-twelfth of the 5\%, or \( \frac{5\%}{12} \), twelve times each year. Since you get this interest twelve times each year, we’ll need to use \( 12t \), rather than \( t \), for the exponent of our function.

The function that represents the amount of money in your savings account \( t \) years after depositing $1000 at 5\% annual interest compounded monthly is 
\[
n(t) = 1000 \left( 1 + \frac{0.05}{12} \right)^{12t}.
\]

Let’s determine the difference between \( m(10) \) and \( n(10) \) from the last two examples. Before we calculate their values, which do you predict will be larger? Since the 5\% annual interest is compounded more often in function \( n \) (twelve times each year) than in function \( m \) (once each year), we should predict that \( n(10) \) is larger than \( m(10) \).

\[
n(10) = 1000 \left( 1 + \frac{0.05}{12} \right)^{12 \cdot 10} \approx 1647.01 \quad \text{and} \quad m(10) = 1000(1.05)^{10} \approx 1628.89
\]

So in ten years we earn about $19 more if the interest is compounded monthly instead of simply.
If an initial deposit of $P_0$ dollars earns interest at the annual rate of $r$ compounded $n$ times each year then the balance $t$ years later is given by the function

$$P(t) = P_0 \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

**EXAMPLE:** Suppose you deposit $50,000 in an account determine how much you have in the account 7 years later if the account pays 3.5% annual interest compounded...

a. ...once each year.

b. ...every day.

**SOLUTIONS:**

a. Let $P_1(t) = 50000\left(1 + \frac{0.035}{1}\right)^t = 50000(1.035)^t$. To answer the question we need to calculate $P_1(7)$:

$$P_1(7) = 50000(1.035)^7$$

$$\approx 63613.96$$

So after 7 years the account's balance is $63,613.96$.

b. Let $P_d(t) = 50000\left(1 + \frac{0.035}{365}\right)^{365t}$. To answer the question we need to calculate $P_d(7)$:

$$P_d(7) = 50000\left(1 + \frac{0.035}{365}\right)^{365\cdot7}$$

$$\approx 63880.32$$

So after 7 years the account’s balance is $63,880.32$. Notice how the same nominal percentage rate produces a greater increase in money when the interest rate is compounded more often.
EXAMPLE: Suppose you invest $1.00 at 100% annual interest. (Yes, this is an unlikely scenario, but sometimes considering extreme situations provides for useful and interesting insight.) Determine the value of the investment after one year if the interest is compounded...

a. …once.

CLICK HERE

b. …quarterly.

CLICK HERE

c. …monthly.

CLICK HERE

d. …daily.

CLICK HERE

e. …hourly.

CLICK HERE

f. …every second.

CLICK HERE

CONCLUSION:

CLICK HERE

In this example, we were introduced to the number $e$. This number is always the same number; it is a constant, not a variable.

\[ e \approx 2.718281828459 \]
The number \( e \) is an *irrational number* which means that it cannot be expressed as the ratio of integers (in other words, \( e \) cannot be expressed as a fraction) and that the decimal expansion for \( e \) never establishes a pattern. Thus, the only way to express the number \( e \) is with the symbol “\( e \)”. We can approximate \( e \) with 2.718281828459 (or with 2.718 or 2.7 if accuracy isn’t important).

The example above implies that, as \( n \) gets very large, \((1 + \frac{1}{n})^n\) gets closer and closer to the number \( e \). We can represent this idea by the symbolic expression below:

\[
\text{As } n \to \infty, \quad \left(1 + \frac{1}{n}\right)^n \to e.
\]

What we mean by this is that you can get the expression \((1 + \frac{1}{n})^n\) as close to \( e \) as you want if you take a large enough value for \( n \).

The discussion above also implies that if you compound interest more and more often, the value of the investment is intimately related to the number \( e \). Note that when interest is compounded “infinitely often” we say that it is *compounded continuously*.

Now we will find a formula analogous to \( P(t) = P_0 \cdot \left(1 + \frac{1}{n}\right)^{n \cdot t} \) but capable of dealing with continuously compounded interest.

Consider \( P(t) = P_0 \cdot \left(1 + \frac{1}{n}\right)^{n \cdot t} \), and let \( n \) get bigger and bigger (i.e., \( n \to \infty \)). Let \( m = \frac{n}{r} \). Then...

\[
P(t) = P_0 \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t} = P_0 \cdot \left(1 + \frac{1}{m}\right)^{m \cdot r \cdot t} = P_0 \cdot \left(1 + \frac{1}{m}\right)^{m \cdot r \cdot t} = P_0 \cdot \left(1 + \frac{1}{m}\right)^{n \cdot t}.
\]

Since \( n \to \infty \) and \( m = \frac{n}{r} \), we can conclude that \( m \to \infty \), so \( \left(1 + \frac{1}{m}\right)^m \to e \), and obtain the formula below.

If an initial deposit of \( P_0 \) dollars earns interest at the annual rate of \( r \) **compounded continuously** then the balance \( t \) years later is given by the function

\[
P(t) = P_0 \cdot e^{rt}
\]
EXAMPLE: Suppose you deposit $1500 in a saving account that earns 4.4% annual interest.

a. Find a function that represents the amount of money in the account $t$ years after making your first deposit if the interest is compounded once a year.

b. Find a function that represents the amount of money in the account $t$ years after making your first deposit if the interest is compounded continuously.

c. In both a and b the annual interest rate is 4.4%. Discuss what makes this rate different in the two cases.

SOLUTIONS:

a. If the interest is compounded simply, then the desired function is $P_s(t) = 1500(1.044)^t$.

b. If the interest is compounded continuously, then the desired function is $P_c(t) = 1500e^{0.044t}$.

c. In both cases, the 4.4% rate is an annual rate, but in a it is a simple annual rate and in b it is a continuous annual rate.

In a, the 4.4% is a simple annual rate, you get interest once at the end of the year, and you get exactly 4.4% of what you put into your account one year prior.

In b, the 4.4% is a continuous annual rate, you get interest every instant, and each instant you get one instant’s worth of a year’s interest. The instant’s worth of 4.4% is calculated based on what is in your account every instant. So once there have been a few instants and you have gotten a few instants worth of interest, your future interest will be calculated based, not just on what you initially deposited in the account, but also on the interest that you have earned in the previous instants. Since you earn “interest on interest” (continuously!), it should be clear that $P_c(t) > P_s(t)$ for all $t > 0$.

For example, 10 years after investing the $1500:

$P_s(10) = 1500(1.044)^{10} \approx 2307.26$

while

$P_c(10) = 1500e^{0.044(10)} \approx 2329.06$

So after 10 years your account would be worth about $22 more if your 4.4% annual interest is compounded continuously instead of simply.
EXAMPLE: Suppose that you invest $7500 at 3.75% annual interest compounded continuously.

a. How much is your investment worth after four years?

b. Use a graph to estimate the *doubling-time* of the investment (i.e., determine how long will take until your investment has doubled).

SOLUTIONS:

a. The function \( P(t) = 7500 \cdot e^{0.0375t} \) models the investment described in the example. To answer the question we need to calculate \( P(4) \):

\[
P(4) = 7500 \cdot e^{0.0375 \cdot 4} \\
\approx 8713.76
\]

So after 4 years your investment’s balance is $8713.76.

b. In order to find the doubling-time, we need find out how long it takes for your initial investment of $7500 to be worth twice as much, or $15,000. So we need to solve the equation \( P(t) = 15000 \):

\[
P(t) = 15000 \\
\Rightarrow 7500 \cdot e^{0.0375t} = 15000 \\
\Rightarrow \frac{7500}{7500} \cdot e^{0.0375t} = \frac{15000}{7500} \\
\Rightarrow e^{0.0375t} = 2
\]

But we don’t yet have the algebraic machinery to solve this equation for \( t \) since it is in the exponent. We need to get \( t \) out of the exponent, and as we’ll see in the next unit, we need logarithmic functions to do this. In the mean time, we can estimate the solutions to this equation by using graphs:

To estimate the solutions to this equation on your graphing calculator, graph the left and right sides of the equation \( e^{0.0375t} = 2 \) separately and observe where they intersect. So, graph \( y_1 = e^{0.0375x} \) and \( y_2 = 2 \) and find the intersection. See Figure 1 below.
According to my calculator, the intersection of $y_1 = e^{0.0375\, x}$ and $y_2 = 2$ occurs at about $(18.5, 2)$ (indicated by the red point on the graph in Figure 1). Thus, the doubling-time of your investment is about 18.5 years. Note that we will be able to solve equations like $e^{0.0375\, t} = 2$ algebraically once we have studied logarithmic functions, which we will do in the next unit.