Section III: Power, Polynomial, and Rational Functions

Unit 1: Power Functions

**DEFINITION:** A **power function** is a function of the form \( f(x) = ax^n \) where \( n \in \mathbb{Z}^{\text{nonneg}} \) (i.e., \( n \) is a nonnegative integer) and \( a \in \mathbb{R} \) (A particular power function will have constants in place of \( a \) and \( n \), leaving \( x \) as the only variable).

**EXAMPLE:** Which of the following functions are power functions? For each power function, state the value of the constants \( a \) and \( n \) in the formula \( y = ax^n \).

\[
\begin{align*}
\text{a. } & b(x) = 5(x - 3)^4 \\
\text{b. } & m(x) = 7 \sqrt[4]{x} \\
\text{c. } & l(x) = 3 \cdot 2^x \\
\text{d. } & s(x) = \sqrt[\frac{7}{x^2}]
\end{align*}
\]

**SOLUTIONS:**

\[
\begin{align*}
\text{a. } & \text{ The function } b(x) = 5(x - 3)^4 \text{ is not a power function because we cannot write it in the form } y = ax^n. \\
\text{b. } & \text{ The function } m(x) = 7 \sqrt[4]{x} \text{ is a power function because we can rewrite its formula as } m(x) = 7 \cdot x^{1/4}. \text{ So } a = 7 \text{ and } n = \frac{1}{4}. \\
\text{c. } & \text{ The function } l(x) = 3 \cdot 2^x \text{ is not a power function because the power is not constant. In fact, } l(x) = 3 \cdot 2^x \text{ is an exponential function.} \\
\text{d. } & \text{ Since } \\
& \sqrt[\frac{7}{x^2}]{x} = \frac{\sqrt{7}}{\sqrt{x^5}} \\
& = \frac{\sqrt{7}}{x^{5/2}} \\
& = \sqrt{7} \cdot x^{-5/2} \\
\text{we see that } s(x) = \sqrt[\frac{7}{x^2}]{x} \text{ can be written in the form } y = ax^n \text{ where } a = \sqrt{7} \text{ and } n = -\frac{5}{2}, \text{ so } s \text{ is a power function.}
\end{align*}
\]
As is the case with linear functions and exponential functions, given two points on the graph of a power function, we can find the function’s formula.

**EXAMPLE:** Suppose that the points $(1, 81)$ and $(3, 729)$ are on the graph of a function $f$. Find an algebraic rule for $f$ assuming that it is …

a. a linear function.  

b. an exponential function  

c. a power function.

**SOLUTIONS:**

**a.** If $f$ is a linear function we know that its rule has form $f(x) = mx + b$. We can use the two given points to solve for $m$.

\[
m = \frac{729 - 81}{3 - 1} = \frac{648}{2} = 324
\]

So now we know that $f(x) = 324x + b$. We can use either one of the given points to find $b$. Let’s use $(1, 81)$:

\[
(1, 81) \Rightarrow f(1) = 81 = 324(1) + b \\
\Rightarrow \quad b = 81 - 324 \\
\Rightarrow \quad b = -243
\]

Thus, if $f$ is linear, its rule is $f(x) = 324x - 243$.

**b.** If $f$ is an exponential function we know its rule has form $f(x) = Ca^x$. We can use the two given points to find two equations involving $C$ and $a$:

\[
(1, 81) \Rightarrow f(1) = 81 = Ca^1 \\
(3, 729) \Rightarrow f(3) = 729 = Ca^3.
\]

In Section II: Unit 2 we solved similar systems of equations by forming ratios. Let’s try a different method here: the *substitution* method.

Let’s start by solving the first equation for $C$:

\[
81 = Ca^1 \\
\Rightarrow \quad C = \frac{81}{a}
\]
Now we can substitute the expression $\frac{81}{a}$ for $C$ in the second equation:

\[
729 = Ca^3
\]

\[
\Rightarrow 729 = \frac{81}{a} \cdot a^3
\]

\[
\Rightarrow 729 = 81 \cdot a^2
\]

\[
\Rightarrow \frac{729}{81} = a^2
\]

\[
\Rightarrow 9 = a^2
\]

\[
\Rightarrow a = \sqrt{9} = 3	ext{ (we don't need } \pm \sqrt{9} \text{ since the base of an exponential function is always positive)}
\]

Now that we know what $a$ is, we can use the fact that $C = \frac{81}{a}$ to find $C$:

\[
C = \frac{81}{a}
\]

\[
= \frac{81}{3}
\]

\[
= 27
\]

Thus, if $f$ is exponential, its rule is $f(x) = 27 \cdot 3^x$.

c. Since $f$ is a power function we know that its rule has form $f(x) = ax^n$. We can use the two given points to find two equations involving $a$ and $n$:

\[
(1, 81) \Rightarrow f(1) = 81 = a(1)^n
\]

\[
(3, 729) \Rightarrow f(3) = 729 = a(3)^n.
\]

We can use the first equation to immediately find $n$.

\[
81 = a(1)^n
\]

\[
\Rightarrow a = 81
\]

Now we can find $n$ by substituting $a = 81$ into the second equation:

\[
729 = 81(3)^n
\]

\[
\Rightarrow \frac{729}{81} = 3^n
\]

\[
\Rightarrow 9 = 3^n
\]

\[
\Rightarrow n = 2	ext{ (note that this could be solved with logarithms if the solution weren't so obvious)}
\]

Thus, if $f$ is a power function, its rule is $f(x) = 81x^2$. 
For a power function $y = ax^n$ the greater the power of $n$, the faster the outputs grow. Below are the graphs of six power functions. Notice that as the power increases, the outputs increase more and more quickly. As $x$ increases without bound (written “$x \to \infty$”), higher powers of $x$ get a lot larger than (i.e., dominate) lower powers of $x$. (Note that we are discussing the long-term behavior of the function.)

As $x$ approaches zero (written “$x \to 0$”) the story is completely different. If $x$ is between 0 and 1, $x^3$ is larger than $x^4$, which is larger than $x^5$. (Try $x = 0.1$ to confirm this). For values of $x$ near zero, smaller powers dominate. On the graph below, notice how on the interval $(0, 1)$ the linear power function $y = x$ dominates power functions of larger power.
EXAMPLE: Use your graphing calculator to graph \( f(x) = 1000x^3 \) and \( g(x) = x^4 \) for \( x > 0 \). Compare the long-term behavior of these two functions.

Could the graphs of \( f(x) = 1000x^3 \) and \( g(x) = x^4 \) intersect again for some value of \( x > 1000 \)? To determine where these graphs intersect, let’s solve the equation \( f(x) = g(x) \):

\[
f(x) = g(x) \\
1000x^3 = x^4 \\
0 = x^4 - 1000x^3 \\
0 = x^3(x - 1000).
\]

Since the only solutions to this equation are \( x = 0 \) and \( x = 1000 \), the graphs of \( f(x) = 1000x^3 \) and \( g(x) = x^4 \) only intersect at \( x = 0 \) and \( x = 1000 \), so they do not intersect when \( x > 1000 \).

EXAMPLE: Use your graphing calculator to graph the power function \( f(x) = x^3 \) and the exponential function \( g(x) = 2^x \) for \( x > 0 \). Compare the long-term behavior of these two functions.

Key Point: Any positive increasing exponential function eventually grows faster than any power function.