The First Derivative Function

Interpreting First Derivative Values

What needs to be communicated when interpreting an instantaneous rate of change for a non-linear function?

- **When?**
  You must convey the value of the input variable at the instant you are describing the rate of change.

- **What?**
  You must communicate whether the output variable is increasing or decreasing at that instant. Also, unless the input variable is "passage of time", you almost always must specify the entity with respect to which the output is changing.

- **How quickly?**
  This part of the description almost always begins with the words "...at a rate of..." Since you have already specified whether the function is increasing or decreasing, the value you state here is always positive. The units on the rate are always "output units per input units."

**Example 1**
The temperature, \( T \), in degrees Fahrenheit, of a cold yam placed in a hot oven is given by \( T = f(t) \), where \( t \) is the time in minutes since the yam was put in the oven.

a. What is the sign of \( f'(t) \)? Why?

The sign of \( f'(t) \) is probably positive since the temperature of the yam is increasing.

b. What are the units of \( f'(20) \)? What is the practical meaning of the statement \( f'(20) = 2 \)?

The units of \( f'(20) \) are \( \frac{\text{°F}}{\text{min}} \). The practical meaning of the statement \( f'(20) = 2 \) is that 20 minutes after the yam was put in the oven, the temperature of the yam was increasing at a rate of 20 \( \frac{\text{°F}}{\text{min}} \).

Note that you cannot say "after 20 minutes." You have to say 20 minutes after the yam was placed in the oven. When we are looking at a derivative at a point, we are looking at a specific point in time or a specific point on a graph. The phrase "after 20 minutes" is not talking about a single point in time. This could be any time after 20 minutes.

c. What is the practical meaning of \( f^{-1}(200) = 5 \)?

The practical meaning of \( f^{-1}(200) = 5 \) is that the temperature of the yam is 200°F, 5 minutes after the yam is placed in the oven.

Note that I included this problem because it is important to recognize the difference between the notation for derivative functions and the notation for inverse functions.
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Example 2

\( M = f(D) \) is the monthly payment ($) on a specific car loan if a down payment \( D \) ($) is made.

What is the meaning of \( f'(2500) = -0.06 \)?

The practical meaning of \( f'(2500) = -0.06 \) is that if the down payment is $2500, the monthly payment is decreasing, with respect to the down payment, at a rate of \( 0.06 \frac{\text{dollars}}{\text{dollar}} \).

Note that it is VERY important to tell what is causing the monthly payment to increase. Whenever the input variable is something other than time, we must do this. To make this doubly confusing, we have dollars for our input and output units, but we are talking about different things.

Leibniz Notation

Another way to write \( f'(a) \) is \( \frac{df}{dx} \bigg|_{x=a} \).

If \( y = f(x) \), another way to write \( f'(x) \) is \( \frac{dy}{dx} \).

Example 3

When making his aerobics tapes, Richard Simmons heart rate increases as the speed of the disco tune increases. \( H = f(M) \) models this relationship where \( H \) is Rick’s heart rate (bpm) and \( M \) is the speed of the tune (bpm). What is the practical meaning of the value \( f'(110) = 0.15 \)?

The practical meaning of \( f'(110) = 0.15 \) is that when the speed of the disco tune is 110 bpm, Richard’s heart rate is increasing, with respect to the speed of the tune, at a rate of \( 0.15 \frac{\text{bpm}}{\text{bpm}} \).
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Example: If the tangent line to \( y = k(x) \) at \((2, 5)\) passes through the point \((1, -3)\), find \(k(2)\) and \(k'(2)\).

\[
\begin{align*}
k(2) & = 5 \quad \text{We were given the point (2,5).} \\

k'(2) & = \frac{\Delta y}{\Delta x} \\
& = \frac{5 - (-3)}{2 - 1} \quad \text{Since we were given two points on the tangent line at } x = 2, \text{ we can find the slope.}
\end{align*}
\]

Example: Sketch the graph of a function \( f \) for which \( f(0) = 3, f'(0) = -2, f''(2) = 0, \) and \( f''(4) = 1 \)

![Figure 12: \( y = f(x) \)](image)

Example: Sketch the graph of a function \( g \) for which \( g(-4) = -5, \ g'(-4) = 3, \ g'(-2) = 0, \ g'(0) = 3, \) and \( g'(3) \) is undefined.

![Figure 13: \( y = g(x) \)](image)