Section 8.5 Graphing Quadratic Equations

The graph of any quadratic equation \( y = ax^2 + bx + c, \ a \neq 0 \) is a parabola. Fill in the following table and then plot the ordered pairs to get an idea of what a parabola looks like.

Table 1: \( y = x^2 - 6x + 8 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 - 6x + 8 )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0^2 - 6\cdot 0 + 8 = 8</td>
<td>(0, 8)</td>
</tr>
<tr>
<td>1</td>
<td>1^2 - 6\cdot 1 + 8 = 1 - 6 + 8 = 3</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>2</td>
<td>2^2 - 6\cdot 2 + 8 = 4 - 12 + 8 = 0</td>
<td>(2, 0)</td>
</tr>
<tr>
<td>3</td>
<td>3^2 - 6\cdot 3 + 8 = 9 - 18 + 8 = -1</td>
<td>(3, -1)</td>
</tr>
<tr>
<td>4</td>
<td>4^2 - 6\cdot 4 + 8 = 16 - 24 + 8 = 0</td>
<td>(4, 0)</td>
</tr>
<tr>
<td>5</td>
<td>5^2 - 6\cdot 5 + 8 = 25 - 30 + 8 = 3</td>
<td>(5, 3)</td>
</tr>
<tr>
<td>6</td>
<td>6^2 - 6\cdot 6 + 8 = 36 - 36 + 8 = 8</td>
<td>(6, 8)</td>
</tr>
</tbody>
</table>

Observations:

The y-intercept is determined by the constant, \( c \), \( (0, c) \)

A parabola is not a line!

Parabolas are symmetrical.

If we fold across the vertical line \( x = 3 \), the two halves of the parabola will lie on top of one another. Mirror image.
Fill in the following table and then graph the parabola.

**Table 2: \( y = x^2 - 6x \)**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 - 6x )</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>((-2)^2 - 6(-2) = 4 + 12 = 16)</td>
<td>((-2, 16))</td>
</tr>
<tr>
<td>-1</td>
<td>((-1)^2 - 6(-1) = 1 + 6 = 7)</td>
<td>((-1, 7))</td>
</tr>
<tr>
<td>0</td>
<td>(0^2 - 6(0) = 0)</td>
<td>((0, 0))</td>
</tr>
<tr>
<td>1</td>
<td>(1^2 - 6(1) = 1 - 6 = -5)</td>
<td>((1, -5))</td>
</tr>
<tr>
<td>2</td>
<td>(2^2 - 6(2) = 4 - 12 = -8)</td>
<td>((2, -8))</td>
</tr>
<tr>
<td>3</td>
<td>(3^2 - 6(3) = 9 - 18 = -9)</td>
<td>((3, -9))</td>
</tr>
</tbody>
</table>

**Opens upwards because \( a > 0 \)**

\( a = 1 \)

---

Where’s the vertex? Where’s the symmetry?

We need to be careful when plotting quadratic equations to make sure we find the vertex and axis of symmetry.
Fill in the following table and then graph the parabola.

Table 3: \( y = -x^2 + 2x \) opens downwards because \( a < 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = -x^2 + 2x )</th>
<th>( (x,y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>Use symmetry</td>
<td>((-2, -8))</td>
</tr>
<tr>
<td>-1</td>
<td>Use symmetry</td>
<td>((-1, -3))</td>
</tr>
<tr>
<td>0</td>
<td>vertex</td>
<td>((0, 0))</td>
</tr>
<tr>
<td>1</td>
<td>Use symmetry</td>
<td>((1, 1))</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>((2, 0))</td>
</tr>
<tr>
<td>3</td>
<td>(-3^2 + 2(3) = -9 + 6 = -3)</td>
<td>((3, -3))</td>
</tr>
<tr>
<td>4</td>
<td>(-4^2 + 2(4) = -16 + 8 = -8)</td>
<td>((4, -8))</td>
</tr>
</tbody>
</table>

The graph of the parabola \( y = 3x^2 - 12x - 15 \) passes through the points \((0, -15)\) and \((4, -15)\).

What is the \( x \)-coordinate of the vertex? How do you know this?

\((0, -15)\) and \((4, -15)\) are symmetrical points on the parabola, so the vertex must lie halfway in between the \( x \)-coordinates at \( x = 2 \).
Find the horizontal-intercepts, axis of symmetry, and vertex of the parabola \( y = x^2 + 4x + 4 \).

\[
0 = x^2 + 4x + 4 \\
0 = (x+2)(x+2)
\]

\[
x + 2 = 0 \\
x + 2 - 2 = 0 - 2 \\
x = -2
\]

There is only one horizontal intercept \((-2, 0)\). So the vertex is the horizontal intercept \((-2, 0)\).

The axis of symmetry is \( x = -2 \).

If there is only one horizontal-intercept, the horizontal-intercept must be the \text{vertex}.

Graph \( y = x^2 + 4x + 4 \).
The **discriminant** of the quadratic equation \( ax^2 + bx + c = 0 \) is the expression \( b^2 - 4ac \). Does this expression look familiar?

It's the radicand of the quadratic formula.

| \( b^2 - 4ac > 0 \) | 2 real number solutions  
| \( b^2 - 4ac = 0 \) | 1 real number solution  
| \( b^2 - 4ac < 0 \) | 0 real number solutions  

The solutions to the equation \( ax^2 + bx + c = 0 \) are the \( x \)-coordinates of the horizontal intercepts of the graph of the parabola \( y = ax^2 + bx + c \).

| \( b^2 - 4ac > 0 \) | 2 \( x \)-intercepts  
| \( b^2 - 4ac = 0 \) | 1 \( x \)-intercept  
| \( b^2 - 4ac < 0 \) | 0 \( x \)-intercepts  

How many horizontal intercepts does the parabola \( y = x^2 + 4x + 3 \) have? \( 2 \)

\[
b^2 - 4ac = 4^2 - 4(1)(3)
= 16 - 12
= 4
\]

How many horizontal intercepts does the parabola \( y = 3x^2 + 7x + 5 \) have? \( 0 \)

\[
b^2 - 4ac = 7^2 - 4(3)(5)
= 49 - 60
= -11
\]

How many horizontal intercepts does the parabola \( y = x^2 - 10x + 25 \) have? \( 1 \)

\[
b^2 - 4ac = (-10)^2 - 4(1)(25)
= 100 - 100
= 0
\]
The \( x \)-coordinate of the vertex of the parabola with equation \( y = ax^2 + bx + c \) is \( x = -\frac{b}{2a} \). This makes sense if you think of the quadratic formula \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) which gives the \( x \)-coordinates of the horizontal intercepts of the parabola with equation \( y = ax^2 + bx + c \) and think about the fact that the \( x \)-coordinate of the vertex is midway between the horizontal intercepts.

Find the vertex of the parabola with equation \( y = -x^2 + 6x + 2 \).

\[
\begin{align*}
\chi &= -\frac{-6}{2(-1)} \\
&= \frac{6}{2} \\
&= 3
\end{align*}
\]

\[
\begin{align*}
y &= -x^2 + 6x + 2 \\
&= -3^2 + 6(3) + 2 \\
&= -9 + 18 + 2 \\
&= 11
\end{align*}
\]

The vertex is \((3, 11)\).

Find the vertex of the parabola with equation \( y = 2t^2 + 7 \).

\[
\begin{align*}
t &= -\frac{0}{2(2)} \\
&= 0
\end{align*}
\]

\[
\begin{align*}
y &= 2t^2 + 7 \\
&= 2(0)^2 + 7 \\
&= 7
\end{align*}
\]

The vertex is \((0, 7)\).
Graph the parabola with equation \( y = -2x^2 + 10x - 15 \)

- **Vertex**
  \[
  x = -\frac{b}{2a} = -\frac{10}{2(-2)} = \frac{5}{2}
  \]
  \[
  y = -2\left(\frac{5}{2}\right)^2 + 10\left(\frac{5}{2}\right) - 15
  \]
  \[
  = -\frac{25}{2} + 25 - 15
  \]
  \[
  = 12.5 - 15
  \]
  \[
  = -2.5
  \]
  Vertex: \((2.5, -2.5)\)

- **Y-intercept**: \((0, -15)\)

- **X-intercepts**: none because the discriminant is negative

- Opens downwards because \( a < 0 \)
  \( a = -2 \)

- \( x = 2 \)
  \[
  y = -2(2)^2 + 10(2) - 15
  \]
  \[
  = -8 + 20 - 15
  \]
  \[
  = -3
  \]

- Table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-27</td>
</tr>
<tr>
<td>0</td>
<td>-15</td>
</tr>
<tr>
<td>1</td>
<td>-7</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>4</td>
<td>-7</td>
</tr>
<tr>
<td>5</td>
<td>-15</td>
</tr>
<tr>
<td>6</td>
<td>-27</td>
</tr>
</tbody>
</table>

- \( x = -1 \)
  \[
  y = -2(-1)^2 + 10(-1) - 15
  \]
  \[
  = -2 - 10 - 15
  \]
  \[
  = -27
  \]

- \( x = 1 \)
  \[
  y = -2(1)^2 + 10(1) - 15
  \]
  \[
  = -2 + 10 - 15
  \]
  \[
  = -7
  \]
The graph of a quadratic equation \( y = ax^2 + bx + c, \ a \neq 0 \) is a **parabola**.

If \( a > 0 \), then the parabola opens **upwards**.  
If \( a < 0 \), then the parabola opens **downwards**.

The vertical intercept of the parabola is the point \((0, c)\).

The first coordinates of the horizontal intercept(s), if any, are solutions to the equation

\[
0 = ax^2 + bx + c
\]

The first coordinate of the vertex, \((h,k)\), can be determined by using the formula \( h = -\frac{b}{2a} \).

The second coordinate of the vertex can be determined by evaluating \( k = ah^2 + bh + c \).

Symmetry may be used to find other solutions to the equation.

Graph \( y = 16 - x^2 \).

The parabola opens **down**.

The vertical intercept is \((0, 16)\).

The horizontal intercepts are \((-4, 0)\) and \((4, 0)\).

The vertex is \((0, 16)\).

\(0\) is halfway between \(-4\) and \(4\).

**Verify:**  
\[
 x = -\frac{b}{2a}
\]

\[
 = - \frac{0}{2(-1)}
\]

\[
 = 0
\]

**horizontal intercepts**

\[
0 = 16 - x^2
\]

\[
0 + x^2 = 16 - x^2 + x^2
\]

\[
x^2 = 16
\]

\[
x = \pm \sqrt{16}
\]

\[
x = \pm 4
\]