The Integral and Comparison Tests
Estimating Sums
In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series
\[ \sum_{k=1}^{\infty} \frac{1}{n(n+1)} \] because in each of those cases we could find a simple formula for the \( n \)th partial sum \( s_n \). But usually it isn’t easy to find \( \lim_{n \to \infty} s_n \).

In this section we develop tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum.
§8.3 The Integral and Comparison Tests: Estimating Sums
Testing with an Integral

Consider: \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \]

There’s no simple formula for the sum \( s_n \) of the first \( n \) terms. The table of values given to the right suggests that the partial sums are approaching a number near 1.64 as \( n \to \infty \) and so it looks as if the series is convergent.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s_n = \sum_{i=1}^{n} \frac{1}{i^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.4636</td>
</tr>
<tr>
<td>10</td>
<td>1.5498</td>
</tr>
<tr>
<td>50</td>
<td>1.6251</td>
</tr>
<tr>
<td>100</td>
<td>1.6350</td>
</tr>
<tr>
<td>500</td>
<td>1.6429</td>
</tr>
<tr>
<td>1000</td>
<td>1.6439</td>
</tr>
<tr>
<td>5000</td>
<td>1.6447</td>
</tr>
</tbody>
</table>
We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve.

**Figure 1**

The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.
§8.3 The Integral and Comparison Tests: Estimating Sums
Testing with an Integral

So the sum of the areas of the rectangles is

\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve \(y = \frac{1}{x^2}\) for \(x \geq 1\), which is the value of the integral \(\int_1^{\infty} \frac{1}{x^2} \, dx\).

The improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than:

\[
\frac{1}{1^2} + \int_1^{\infty} \frac{1}{x^2} \, dx = 2
\]

Thus the partial sums are bounded and the series converges. The sum of the series (the limit of the partial sums) is also less than 2.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots < 2
\]
§8.3 The Integral and Comparison Tests: Estimating Sums

Testing with an Integral

Consider: \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \)

The table of values of \( s_n \) suggests that the partial sums aren’t approaching a finite number, so we suspect that the given series may be divergent.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.2317</td>
</tr>
<tr>
<td>10</td>
<td>5.0210</td>
</tr>
<tr>
<td>50</td>
<td>12.7524</td>
</tr>
<tr>
<td>100</td>
<td>18.5896</td>
</tr>
<tr>
<td>500</td>
<td>43.2834</td>
</tr>
<tr>
<td>1000</td>
<td>61.8010</td>
</tr>
<tr>
<td>5000</td>
<td>139.9681</td>
</tr>
</tbody>
</table>

Recall: \( I_p = \int_{1}^{\infty} \frac{1}{x^p} \, dx \)

\( I_1 = \int_{1}^{\infty} \frac{1}{x} \, dx \)

\( I_2 = \int_{1}^{\infty} \frac{1}{x^2} \, dx \)
§8.3 The Integral and Comparison Tests: Estimating Sums

Testing with an Integral

Again we use a picture for confirmation. Figure 2 shows the curve but this time we use rectangles whose tops lie above the curve.

![Figure 2](image)

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the left endpoint of the interval.
§8.3 The Integral and Comparison Tests: Estimating Sums

Testing with an Integral

So the sum of the areas of the rectangles is
\[
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}
\]

This total area is greater than the area under the curve \( y = 1/\sqrt{x} \) for \( x \geq 1 \), which is the value of the integral \( \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \). This improper integral diverges. Both the picture and the integral indicates that the series is divergent.
The same sort of geometric reasoning used for these two series can be used to prove The Integral Test.

**The Integral Test** Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

(a) $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

(b) $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent
§8.3 The Integral and Comparison Tests: Estimating Sums

Using The Integral Test - Example

Determine if the series \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \) converges or diverges.

\[ a_n = \frac{\ln(n)}{n} \quad , \quad \lim_{n \to \infty} a_n = \frac{\ln(n)}{n} = 0 \]

\[ f(x) = \frac{\ln(x)}{x} \quad \Rightarrow \quad \frac{df}{dx} = \frac{1 - \ln(x)}{x^2} < 0 \]

\[ I = \int_{1}^{\infty} \ln(x) \frac{1}{x} \, dx \]

\[ I = \int_{1}^{\infty} u \, du \]

\[ = \left[ \frac{u^2}{2} \right]_{1}^{\infty} \]

\[ = \frac{u^2}{2} \bigg|_{1}^{\infty} \]

let \( u = \ln(x) \)

\[ du = \frac{1}{x} \, dx \]

\[ u(1) = 0 \]
The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a $p$-series.  

The $p$-series is convergent if $p > 1$ and divergent if $p \leq 1$.

The harmonic series: $\sum_{k=1}^{\infty} \frac{1}{k}$ is a $p$-series with $p = 1$ and is divergent.

The series: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent because it is a $p$-series with $p > 1$. 
\[ I_1 = \int_1^\infty \frac{1}{x} \, dx = \ln(x) \bigg|_1^\infty \]

\[ I_2 = \int_1^\infty \frac{1}{x^2} \, dx = \int_1^\infty x^{-2} \, dx = \frac{1}{x} \bigg|_1^\infty \]

\[ I_{-1} = \int_{-1}^\infty \frac{1}{1} \, dx \quad \text{Improper integral diverges} \]
The series $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ which is a geometric series $\sum_{n=1}^{\infty} a \cdot r^{n-1}$ with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent.

Because the series (2) is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$a_n = \frac{1}{2^n + 1} < \frac{1}{2^n} = b_n$$

shows that our given series (2) has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series).
This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

\[ \sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1 \]

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive.
The Comparison Test

Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms.

(a) If \( \sum b_n \) is convergent and \( a_n \leq b_n \) for all \( n \), then \( \sum a_n \) is also convergent.

(b) If \( \sum b_n \) is divergent and \( a_n \geq b_n \) for all \( n \), then \( \sum a_n \) is also divergent.

The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.
In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A $p$-series [$\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]
- A geometric series [$\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]
Determine convergence/divergence of the series:

\[
\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}
\]

\[\Rightarrow a_n = \frac{5}{2n^2 + 4n + 3}\]

\[
\sum_{n=1}^{\infty} \frac{5}{2n^2}
\]

converges

\[
\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}
\]

converges
The Limit Comparison Test  Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

Where $c > 0$ is a finite positive number, then either both series converge or both diverge.

Example: Determine if the series $\sum_{n=1}^{\infty} \frac{1}{2n^3 + 1}$ converges or diverges by comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

$$A_n = \frac{1}{2n^3 + 1}, \quad B_n = \frac{1}{n^3}$$

$$\lim_{n \to \infty} \frac{A_n}{B_n} = \lim_{n \to \infty} \frac{1}{2n^3 + 1} \cdot \frac{n^3}{1} = \lim_{n \to \infty} \frac{n^3}{2n^3 + 1} = \lim_{n \to \infty} \frac{1}{2}$$
Suppose we have been able to use the Integral Test to show that a series $\sum a_n$ is convergent and we now want to find an approximation to the sum $s$ of this series.

Of course, any partial sum $s_n$ is an approximation to $s$ because $\lim_{n \to \infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder $R_n$ is the error made when $s_n$, the sum of the first $n$ terms, is used as an approximation to the total sum.
§8.3 The Integral and Comparison Tests: Estimating Sums
Estimating the Sum of a Series

We use the same notation and ideas as in the Integral Test, assuming that $f$ is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in Figure 3, we see that:

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)dx$$

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)dx$$
3 Remainder Estimate for the Integral Test  Suppose $f(k) = a_k$, where $f$ is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx$$

*Note: lower limit of integration.*
8.3 The Integral and Comparison Tests: Estimating Sums
The Remainder Estimate Example Computation

a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

b) How many terms are required to ensure that the sum is accurate to within 0.0005?

In both cases, we need $\int_0^\infty \frac{1}{x^3} \, dx = \frac{1}{2} \times \frac{1}{2}$

\[
a) \quad \sum_{n=1}^{10} \frac{1}{n^3} \times S_{10} = \frac{1}{2} \times 10^2 = \frac{1}{200} = 0.0005
\]

\[
b) \quad R_n = 0.0005
\]

\[
R_n \leq \int_n^\infty \frac{1}{x^3} \, dx \leq 0.0005
\]
§8.3 The Integral and Comparison Tests: Estimating Sums

Estimating the Sum of a Series

We use the same notation and ideas as in the Integral Test, assuming that \( f \) is decreasing on \([n, \infty)\). Comparing the areas of the rectangles with the area under \( y = f(x) \) for \( x > n \), we see that:

\[
R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)dx
\]

Similarly, we see that

\[
R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)dx
\]
Present the analysis to determine whether the following series converge or diverge.

1. \[ \sum_{k=1}^{\infty} \frac{1}{2k^2 + k} \]

2. \[ \sum_{k=1}^{\infty} \frac{3^{1+2k}}{k^{613} + 5^k} \]

3. \[ \sum_{k=1}^{\infty} \frac{\cos^2(k)}{k\sqrt{k}} \]

4. \[ \sum_{k=1}^{\infty} \frac{\ln(k)}{\sqrt{k}} \]