§5.2 The Definite Integral
Definition and Lexicon

2. Definition: If $f$ is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a,b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a)$, $x_1$, ..., $x_{n-1}$, $x_n (= b)$ be the endpoints of these subintervals and we choose sample points $x_1^*$, $x_2^*$, ..., $x_{n-1}^*$, $x_n^*$ in these subintervals, so $x_j^*$ lies in the $j$th subinterval $[x_{j-1}, x_j]$. Then the definite integral of $f$ from $a$ to $b$ is:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j^*) \Delta x$$

- $\sum_{j=1}^{n} f(x_j^*) \Delta x$ is called a Riemann sum.
- The elongated $S$ is called the integral sign.
- The function $f(x)$ is called the integrand.
- The values $a$ and $b$ are called the lower and upper limits of the integral.
- The procedure for calculating the integral is called integration.
§5.2 The Definite Integral

Definite Integral – Geometric Interpretation if $f(x) \geq 0$

If $f(x) \geq 0$, $a \leq x \leq b$ and

$$
\lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x = \int_{a}^{b} f(x) \, dx
$$

If $f(x) \geq 0$, $a \leq x \leq b$ and

$$
\lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(x_j) \Delta x = \int_{a}^{b} f(x) \, dx
$$

$y = f(x)$

$$
\int_{a}^{b} f(x) \, dx = \text{Area under curve } y = f(x), \ a \leq x \leq b
$$

Left Riemann Sum = Sum of Areas of Rectangles.

Right Riemann Sum = Sum of Areas of Rectangles.
§5.2 The Definite Integral

Definite Integral – Geometric Interpretation, General Case

If $f(x)$ takes both positive and negative values:

\[
\int_a^b f(x)\,dx = A_1 + A_2 + A_3
\]

N.B.: The integral can be interpreted as the difference between the area bounded by the horizontal axis and $f(x)$, when $f(x) > 0$ (i.e., $A_1$ and $A_3$), and the area bounded by horizontal axis and $f(x)$, when $f(x) < 0$,(i.e., $-A_2$).
§5.2 The Definite Integral
Evaluating Integrals – Example Summations

\[ \sum_{j=1}^{n} 1 = 1 + 1 + \ldots + 1 = n \]

\[ \sum_{j=1}^{n} j = 1 + 2 + 3 + \ldots + n-1, n = \frac{n(n+1)}{2} \]

\[ S = 1 + 2 + 3 + \ldots + 99 + 100 \]

\[ S = 100 + 99 + 98 + \ldots + 2 + 1 \]

\[ 2S = 101 + 101 + \ldots + 101 + 101 \]

\[ S = \frac{100 \cdot 101}{2} \]
§5.2 The Definite Integral
Evaluating Integrals – Some Pretty Summations

\[ \sum_{j=1}^{n} 1 = n = 1 + 1 + 1 + 1 + \ldots \]

\[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} = 1 + 2 + 3 + \ldots + n - 1 + n \]

\[ \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} = 1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2 + n^2 \]

\[ \sum_{j=1}^{n} j^3 = \left[ \frac{n(n+1)}{2} \right]^2 = 1^3 + 2^3 + 3^3 + \ldots + (n - 1)^3 + n^3 \]
§5.2 The Definite Integral
Summation Properties

For constants $a_j$, $b_j$, and $c$:

- $\sum_{j=1}^{n} c = nc$
- $\sum_{j=1}^{n} ca_j = c \sum_{j=1}^{n} a_j = c (a_1 + a_2 + \cdots + a_n)$
- $\sum_{j=1}^{n} (a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j$
- $\sum_{j=1}^{n} (a_j - b_j) = \sum_{j=1}^{n} a_j - \sum_{j=1}^{n} b_j$

E.g.: $c = \pi$, $n = 4$,
\{a_1, a_2, a_3, a_4\} = \{1, 3, 5, 7\},
\{b_1, b_2, b_3, b_4\} = \{0, 2, 4, 6\}$

\[
\Rightarrow \sum_{j=1}^{4} \pi = \pi + \pi + \pi + \pi = 4\pi
\]
§5.2 The Definite Integral
Evaluating Riemann Sums

(a) Evaluate the right Riemann sum of \( f(x) = x^3 - 6x \), with \( a = 0 \), \( b = 3 \), and \( n = 6 \).

\[
\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{3}{6} = \frac{1}{2}
\]

\[
x_j = a + j \cdot \Delta x = 0 + j \cdot \frac{1}{2}
\]

\[
\sum_{j=1}^{n} j = \frac{n(n+1)}{2}
\]

\[
\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}
\]

\[
\sum_{j=1}^{n} j^3 = \left( \frac{n(n+1)}{2} \right)^2
\]
\[ R_n = \frac{3^4}{n^4} \sum_{j=1}^{n} j^3 - \frac{6 \cdot 3^2}{n^2} \sum_{j=1}^{n} j \]

\[ = \frac{3^4}{n^4} \left( \frac{n(n+1)}{2} \right)^2 - \frac{6 \cdot 3^2}{n^2} \frac{n(n+1)}{2} \]

\[ = \frac{3^4}{4} \frac{n^2(n+1)^2}{n^4} - \frac{6 \cdot 3^2}{2} \frac{n(n+1)}{n^2} \]

\[ R_n = \frac{3^4}{4} \left( 1 + \frac{1}{n} \right)^2 - \frac{6 \cdot 3^2}{2} \left( 1 + \frac{1}{n} \right) \]

\[ P_b = \frac{3^4}{4} \left( 1 + \frac{1}{6} \right)^2 - \frac{6 \cdot 3^2}{2} \left( 1 + \frac{1}{6} \right) \]

\[ \lim_{n \to \infty} R_n = \frac{3^4}{4} - \frac{6 \cdot 3^2}{2} \]
(a) Evaluate the right Riemann sum of \( f(x) = x^3 - 6x \), with \( a = 0 \), \( b = 3 \), and \( n = 6 \).

\[
R_6 = \sum_{j=1}^{6} f(0 + j\Delta x)\Delta x \\
\Delta x = \frac{3 - 0}{6} = \frac{1}{2} \\
x_j = a + j\Delta x = 0 + j \frac{3}{6} = \frac{j}{2}
\]

\[
R_6 = \sum_{j=1}^{6} f\left(\frac{j}{2}\right) \cdot \frac{1}{2} = \frac{1}{2} \sum_{j=1}^{6} f\left(\frac{j}{2}\right)
\]

\[
= \frac{1}{2} \left(f\left(\frac{1}{2}\right) + f\left(\frac{2}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{6}{2}\right)\right)
\]

\[
\approx -3.9375
\]
(b) Evaluate \( \int_{a}^{b} f(x) \, dx \) where \( f(x) = x^3 - 6x \), with \( a = 0 \) and \( b = 3 \).

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x
\]
§5.2 The Definite Integral
Evaluating Integrals – Solution

(b) Evaluate \( \int_{a}^{b} f(x) \, dx \) where \( f(x) = x^3 - 6x \), with \( a = 0 \) and \( b = 3 \).

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x
\]

\[
= \lim_{n \to \infty} \sum_{j=1}^{n} f \left( \frac{3j}{n} \right) \left( \frac{3}{n} \right)
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \sum_{j=1}^{n} \left( \left( \frac{3j}{n} \right)^3 - 6 \left( \frac{3j}{n} \right) \right)
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \sum_{j=1}^{n} \left( \left( \frac{3}{n} \right)^3 j^3 - \left( \frac{18}{n} \right) j \right)
\]

\[
\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}
\]

\[
x_j = a + j \Delta x = 0 + j \frac{3}{n} = j \frac{3}{n}
\]
(b) Evaluate \( \int_{a}^{b} f(x) \, dx \) where \( f(x) = x^3 - 6x \), with \( a = 0 \) and \( b = 3 \).

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \sum_{j=1}^{n} \left( \left( \frac{3}{n} \right)^3 j^3 - \left( \frac{18}{n} \right) j \right)
\]

\[
= \lim_{n \to \infty} \left[ \frac{81}{n^4} \sum_{j=1}^{n} j^3 - \frac{54}{n^2} \sum_{j=1}^{n} j \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{81}{n^4} \left( \frac{n(n+1)}{2} \right)^2 - \frac{54}{n^2} \left( \frac{n(n+1)}{2} \right) \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{81}{n^4} \left( \frac{n^2 + 2n + 1}{4} \right)^2 - \frac{54}{n^2} \left( \frac{n^2 + n}{2} \right) \right]
\]
§5.2 The Definite Integral
Evaluating Integrals – Interpretation

(b) Evaluate \( \int_{a}^{b} f(x) \, dx \) where \( f(x) = x^3 - 6x \), with \( a = 0 \) and \( b = 3 \).

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x
\]

\[
= \lim_{n \to \infty} \left[ \frac{81}{4} \left(1 + \frac{1}{n}\right)^2 - 27 \left(1 + \frac{1}{n}\right) \right]
\]

\[
= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75
\]

This result **cannot** be interpreted as the area bounded by \( f(x) \) and the horizontal axis.

This result can be interpreted as the difference between \( A_2 \), the area bounded by \( f(x) \) and the \( x \)-axis for \( f(x) < 0 \) and \( A_1 \), the area bounded by \( f(x) \) and the \( x \)-axis for \( f(x) > 0 \).
§5.2 The Definite Integral

Midpoint Rule

\[
\int_a^b f(x)dx \approx \sum_{j=1}^{n} f(x_j^*) \Delta x \\
\text{where } \Delta x = \frac{b-a}{n}
\]

\[
\int_a^b f(x)dx \approx \sum_{j=1}^{n} f(\bar{x}_j) \Delta x, \quad \bar{x}_j = \left( x_{j-1} + x_j \right)/2 \quad = \text{midpoint of } [x_{j-1}, x_j]
\]
§5.2 The Definite Integral
Midpoint Rule – Example Computation

Use the Midpoint Rule with $n = 10$ to approximate the integral: $\int_{1}^{2} \sqrt{1 + x^2} \, dx$ rounded to four decimal places.

\[ a = 1, \quad b = 2, \quad \Delta x = \frac{b-a}{n} = \frac{1}{10} \]

\[ x_j = a + j\Delta x \]

\[ \bar{x}_j = \frac{x_{j-1} + x_j}{2} = \frac{a + (j-1)\Delta x + a + j\Delta x}{2} = a + (j - \frac{1}{2})\Delta x \]

\[ M_n = \sum_{j=1}^{n} f\left(\bar{x}_j\right)\Delta x = \sum_{j=1}^{n} f\left(a + (j - \frac{1}{2})\right)\Delta x \]

\[ = \sum_{j=0}^{n-1} f\left(a + (j + \frac{1}{2})\right)\Delta x \]
§5.2 The Definite Integral
Properties of Integrals

For some constants $a$, $b$, and $c$:

1. \[ \int_{a}^{b} cdx = c(b - a) \]

2. \[ \int_{a}^{b} cf(x)dx = c \int_{a}^{b} f(x)dx \]

3. \[ \int_{a}^{b} [f(x) + g(x)]dx = \int_{a}^{b} f(x)dx + \int_{a}^{b} g(x)dx \]

4. \[ \int_{a}^{b} [f(x) - g(x)]dx = \int_{a}^{b} f(x)dx - \int_{a}^{b} g(x)dx \]

5. \[ \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx = \int_{a}^{b} f(x)dx, \quad a \leq c \leq b \]
§5.2 The Definite Integral
Comparison Properties of Integrals

6. \( f(x) \geq 0 \) for \( a \leq x \leq b \),
\[
\Rightarrow \int_{a}^{b} f(x) \, dx \geq 0
\]

7. \( f(x) \geq g(x) \) for \( a \leq x \leq b \),
\[
\Rightarrow \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx
\]

8. \( m \leq f(x) \leq M \) for \( a \leq x \leq b \),
\[
\Rightarrow m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a)
\]