The area of a rectangle is the product of the length and the width.

The area of a triangle is half the base times the height.

The area of a polygon could be found by dividing it into triangles and adding the areas of the triangles.

\[
A = lw
\]

\[
A = \frac{1}{2}bh
\]

\[
A = A_1 + A_2 + A_3 + A_4
\]
Let $S$ be the region bounded by $y = f(x)$, $y = 0$ (the horizontal axis), $x = a$ and $x = b$.

We might approximate the area of $S$, loosely referred to as the area "under" the curve, by breaking $S$ into smaller rectangles then summing the areas.

\[
A_1 = f(x_1) (x_1 - x_0) = f(x_1) \Delta x \\
A_2 = f(x_2) (x_2 - x_1) = f(x_2) \Delta x \\
A_3 = f(x_3) (x_3 - x_2) = f(x_3) \Delta x \\
A_4 = f(x_4) (x_4 - x_3) = f(x_4) \Delta x
\]

Then $A$, the "area" of the bounded region is approximated as:

\[
A \approx A_1 + A_2 + A_3 + A_4 = \sum_{j=1}^{4} A_j
\]
§5.1 Areas & Distances

The Right Riemann Sum – Sigma Notation

Let $R_n$ be the approximate the area “under” the curve for $x \mid a \leq x \leq b$ over $n$ subintervals.

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x$$

where: $\Delta x = \frac{b-a}{n}$, $x_j = a + j\Delta x$

For the Right Riemann Sum:

$j = 1 \Rightarrow x_1 = x_0 + 1 \cdot \Delta x$

$j = 2 \Rightarrow x_2 = x_0 + 2 \cdot \Delta x$

$j = 3 \Rightarrow x_3 = x_0 + 3 \cdot \Delta x$

\vdots

$j = n \Rightarrow x_n = x_0 + n \cdot \Delta x$
§5.1 Areas & Distances

A, The Area “Under” the Curve from the Right Riemann Sum

\[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x \]

When \( f(x) \) is increasing
\( R_n \) is an overestimate

\[ \Delta x = \lim_{n \to \infty} \frac{b-a}{n} \]

When \( f(x) \) is decreasing
\( R_n \) is an underestimate

Definition: The "area" \( A \) of the region \( S \) that lies "under" the graph of the continuous function \( f \) is the limit of the sum of the areas of approximating rectangles:

\[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x \]
§5.1 Areas & Distances

The Left Riemann Sum – Sigma Notation

Let $L_n$ be the approximate the area “under” the curve for $x \mid x_0 \leq x \leq x_n$ over $n$ subintervals.

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x$$

where: $\Delta x = \frac{b-a}{n}$, $x_{j-1} = a + (j-1)\Delta x$

For the Left Riemann Sum:

- $j = 1 \Rightarrow x_0 = x_0 + (1-1) \cdot \Delta x$
- $j = 2 \Rightarrow x_1 = x_0 + (2-1) \cdot \Delta x$
- $j = 3 \Rightarrow x_2 = x_0 + (3-1) \cdot \Delta x$
- $\vdots$
- $j = n \Rightarrow x_{n-1} = x_0 + (n-1) \cdot \Delta x$
§5.1 Areas & Distances

The Left Riemann Sum – Alternative Sigma Notation

Let $L_n$ be the approximate area “under” the curve for $x \mid x_0 \leq x \leq x_n$ over $n$ subintervals.

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x$$

where: $\Delta x = \frac{b-a}{n}$, $x_j = a + j\Delta x$

For the Left Riemann Sum:

1. $j = 0 \Rightarrow x_0 = x_0 + 0 \cdot \Delta x$
2. $j = 1 \Rightarrow x_1 = x_0 + 1 \cdot \Delta x$
3. $j = 2 \Rightarrow x_2 = x_0 + 2 \cdot \Delta x$
4. \vdots
5. $j = n-1 \Rightarrow x_{n-1} = x_0 + (n-1) \cdot \Delta x$
§5.1 Areas & Distances

A, The Area “Under” the Curve from the Left Riemann Sum

\[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j)\Delta x \]

When \( f(x) \) is increasing, \( L_n \) is an underestimate.

When \( f(x) \) is decreasing, \( L_n \) is an overestimate.

Definition: The "area" \( A \) of the region \( S \) that lies "under" the graph of the continuous function \( f \) is the limit of the sum of the areas of approximating rectangles:

\[ A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(x_j)\Delta x \]

\[ \Delta x = \lim_{n \to \infty} \frac{b-a}{n} \]
§5.1 Areas & Distances

A, The Area “Under” the Curve as the Limiting Sum

\[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x \]

\[ A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(x_j) \Delta x \]

\[ A = \lim_{n \to \infty} \sum_{j=1}^{n} f(x^*_j) \Delta x \]

**Definition:** The "area" \( A \) of the region \( S \) that lies "under" the graph of the continuous function \( f \) is the limit of the sum of the areas of approximating rectangles:

\[ A = \lim_{n \to \infty} \sum_{j=1}^{n} f(x^*_j) \Delta x \]
Use $R_n$, the right Riemann Sum to approximate $A$, the area of the region bounded by $y = 0$ and $y = \cos(x)$ on the interval $0 \leq x \leq \pi/2$, rounded to six decimal places. Use $n$ subintervals, with $n = 10, 50, 100$. 

\[
R_n = \sum_{j=1}^{n} f(x_j) \Delta x \\
= \sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) \frac{\pi}{2n} \\
= \cos\left(\frac{\pi}{2n}\right) \frac{\pi}{2n} + \cos\left(\frac{2\pi}{2n}\right) \frac{\pi}{2n} + \cos\left(\frac{3\pi}{2n}\right) \frac{\pi}{2n} + \ldots
\]
Use $R_n$, the right Riemann Sum to approximate $A$, the area of the region bounded by $y = 0$ and $y = \cos(x)$ on the interval $0 \leq x \leq \pi/2$, rounded to six decimal places. Use $n$ subintervals, with $n = 10, 50, 100$.

$$R_n = \sum_{j=1}^{n} f(x_j) \Delta x, \quad \text{where} \quad \Delta x = \frac{b-a}{n}, \quad x_j = a + j\Delta x$$

$$= \sum_{j=1}^{n} f(a + j\Delta x) \Delta x,$$

$$= \sum_{j=1}^{n} \left[ f \left( a + j \left( \frac{b-a}{n} \right) \right) \cdot \left( \frac{b-a}{n} \right) \right]$$

$$= \sum_{j=1}^{n} \left[ \cos \left( 0 + j \left( \frac{\pi/2}{n} \right) \right) \cdot \left( \frac{\pi/2}{n} \right) \right]$$
§5.1 Areas & Distances
Area Estimation via Uncle Riemann – Casio Commands

Use $R_n$, the right Riemann Sum to approximate $A$, the area of the region bounded by $y = 0$ and $y = \cos(x)$ on the interval $0 \leq x \leq \pi/2$, rounded to six decimal places. Use $n$ subintervals, with $n = 10, 50, 100$.

Define $rr(n) = \sum (\cos(\frac{k\pi}{2n})) \times \pi/(2n), k, 1, n$
§5.1 Areas & Distances

Area Estimation via Uncle Riemann – Casio Results and Questions

Use $R_n$, the right Riemann Sum to approximate $A$, the area of the region bounded by $y = 0$ and $y = \cos(x)$ on the interval $0 \leq x \leq \pi/2$, rounded to six decimal places. Use $n$ subintervals, with $n = 10, 50, 100$.

Conclusions:

$A \approx R_{10} = 0.919403$

$A \approx R_{50} = 0.984210$

$A \approx R_{100} = 0.992125$

Questions:

a) Does $R_n$ underestimate or overestimate the true value of $A$? Explain.

b) Predict: $A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j)\Delta x$
§5.1 Areas & Distances
Area Estimation via Uncle Riemann – TI89 Commands

Use $L_n$, the left Riemann Sum to approximate $A$, the area of the region bounded by $y = 0$ and $y = \cos(x)$ on the interval $0 \leq x \leq \pi/2$, rounded to six decimal places. Use $n$ subintervals, with $n = 10, 50, 100$.

Define $lr(n) = \sum (\cos\left(\frac{k \pi}{2n}\right) \times \frac{\pi}{2n}, k, 0, n-1)$
§5.1 Areas & Distances

Area Estimation via Uncle Riemann – TI89 Results and Questions

Use $L_n$, the left Riemann Sum to approximate $A$, the area of the region bounded by $y = 0$ and $y = \cos(x)$ on the interval $0 \leq x \leq \pi/2$, rounded to six decimal places. Use $n$ subintervals, with $n = 10, 50, 100$.

Conclusions:

$A \approx L_{10} = 1.076483$

$A \approx L_{50} = 1.015626$

$A \approx L_{100} = 1.007833$

Questions:

a) Does $L_n$ underestimate or overestimate the true value of $A$? Explain.

b) Predict: $A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(x_j) \Delta x$
Suppose the odometer of your car is broken so you take speedometer readings every five seconds and record these in Table 1. Use these data to estimate the distance you have driven over the 30-second time interval.

<table>
<thead>
<tr>
<th>$t \sim \text{sec}$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v \sim \text{mi/hr}$</td>
<td>17</td>
<td>21</td>
<td>24</td>
<td>29</td>
<td>32</td>
<td>31</td>
<td>28</td>
</tr>
<tr>
<td>$v \sim \text{ft/sec}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
§5.1 Areas & Distances
Left and Right Riemann Sums

\[ R_n = \sum_{j=1}^{n} f(x_j) \Delta x, \quad \text{where} \quad \Delta x = \frac{b-a}{n}, \quad x_j = a + j\Delta x \]

\[ L_n = \sum_{j=0}^{n-1} f(x_j) \Delta x, \quad \text{where} \quad \Delta x = \frac{b-a}{n}, \quad x_j = a + j\Delta x \]

\[ R_n = \sum_{j=1}^{n} f(x_j) \Delta x = [f(x_1) + f(x_2) + \cdots + f(x_{n-2}) + f(x_{n-1}) + f(x_n)] \Delta x \]

\[ L_n = \sum_{j=0}^{n-1} f(x_j) \Delta x = [f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-2}) + f(x_{n-1})] \Delta x \]

\[ R_n = -f(x_0) \Delta x + L_n + f(x_n) \Delta x \]

\[ L_n = f(x_0) \Delta x + R_n - f(x_n) \Delta x \]