An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is:

$$F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where $\mu$ is a positive constant called the coefficient of friction and where $0 \leq \theta \leq \pi/2$. Find the value of $\theta$ that will optimise the force (in this case, we want the angle $\theta$ that will minimise $F$).
§4.6 Optimisation
Applied Extrema – A Solution Strategy

1. **Understand the Problem.** The first step is to read the problem carefully until it clearly understood. Ask yourself:
   - What is the unknown?
   - What are the given quantities and associated units?
   - What are the given conditions?
   - *Are there any assumptions implied in the problem statement?*

2. **Draw a Cartoon.** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. **Introduce Notation.** Assign a symbol to the quantity that is to be maximised or minimised (let us call it $Q$ for now.) Also select symbols ($a, b, c, ... x, y$) for other quantities and label the diagram with these symbols. It may help to use suggestive symbols - *e.g.*, $A$ for area, $h$ for height, $t$ for time and to note all appropriate units.

4. **Find Expression for $Q$** in terms of some of the other symbols from Step 3.
5. If $Q$ has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for $Q$. Thus, $Q$ will be expressed as a function of one variable $x$, say $Q = f(x)$. Write the domain of this function.

6. **Find the Extrema.** Use the methods on §4.2 and §4.3 to find the absolute extrema of $f$. In particular, if the domain of $f$ is a closed interval, then the Closed Interval Method in §4.2 can be used.

7. **Sanity Check.** Are magnitudes reasonable? Are units consistent?

8. **Conclusion.** Make a clear, concise statement of the analysis conclusion(s). Read the problem statement again to ensure that the conclusions are consistent with the problem statement.
An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is:

$$F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)} = \frac{\mu W}{B(\theta)}$$

where $\mu$ is a positive constant called the coefficient of friction and where $0 \leq \theta \leq \pi/2$. Find the value of $\theta$ that will optimise the force (in this case, we want the angle $\theta$ that will minimise $F$).
\( B(\theta) = \mu \sin(\theta) + \cos(\theta) \)

\( \frac{dB}{d\theta} = \mu \cos(\theta) - \sin(\theta) = 0 \iff \mu \cos(\theta) = \sin(\theta) \)

\( \mu = \frac{\sin(\theta)}{\cos(\theta)} \)

\( = \tan(\theta) \)

\( \arctan(\mu) = \theta_0 \)

\( \frac{d^2B}{d\theta^2} = -\mu \sin(\theta) - \cos(\theta) < 0 \text{ for } 0 < \theta < \frac{\pi}{2} \)
\[ \therefore \theta_c = \arctan(\mu) \]

\[ F_{\text{min}} = F(\theta_c) \]
An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is:

$$F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where $\mu$ is a positive constant called the coefficient of friction and where $0 < \theta < \pi/2$. Find the value of $\theta$ that will optimise the force (in this case, we want the angle $\theta$ that will minimise $F$).

1. **Understand the Problem.**

   We wish to find the angle $\theta \mid 0 \leq \theta \leq \pi/2$, which will minimise $F$, the force required to drag the object. Here, $W$ and $\mu$ are positive constants, so maximising the denominator of $F$, call it $B(\theta) = \mu \sin(\theta) + \cos(\theta)$, will minimise $F$. 

An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is:

$$F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where $\mu$ is a positive constant called the *coefficient of friction* and where $0 < \theta < \pi/2$. Find the value of $\theta$ that will optimise the force (in this case, we want the angle $\theta$ that will minimise $F$).

2. **Draw a Cartoon.**

We wish to find the angle $\theta$, $0 \leq \theta \leq \pi/2$, which will minimise $F$, the force required to drag the object. Here, $W$ and $\mu$ are positive constants, so maximising the denominator of $F$, call it $B(\theta) = \mu \sin(\theta) + \cos(\theta)$, will minimise $F$.
6. **Find the Extrema.**

To minimise:

\[ F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}, \quad 0 \leq \theta \leq \pi / 2 \]

we seek the value(s) of \( \theta \) that maximises \( B(\theta) = \mu \sin(\theta) + \cos(\theta) \).  

- Find the critical number(s)

\[
\frac{dB}{d\theta} = \mu \cos(\theta) - \sin(\theta) = 0 \quad \Rightarrow \quad \mu = \frac{\sin(\theta)}{\cos(\theta)} \quad \Rightarrow \quad \theta_C = \arctan(\mu)
\]

where \( \theta_C \) is the critical number.
• Characterise $\theta_C$, i.e., determine if the function at $\theta_C$ is a maximum, minimum, or neither.

$B'$ is continuous at $\theta_C$ so, from the **Second Derivative Test**,  
\[(a)\] If $B'(\theta_C) = 0$ and $B''(\theta_C) > 0$, then $B$ has a local minimum at $\theta_C$.  
\[(b)\] If $B'(\theta_C) = 0$ and $B''(\theta_C) < 0$, then $B$ has a local maximum at $\theta_C$.

\[
\frac{dB}{d\theta} = \mu \cos(\theta) - \sin(\theta)
\]
\[
\frac{d^2 B}{d\theta^2} = -\mu \sin(\theta) - \cos(\theta)
\]

We are given $0 \leq \theta \leq \pi/2$, which means that $\sin(\theta) \geq 0$ and $\cos(\theta) \geq 0$. Because $\mu \geq 0$ we have $B''(\theta_C) < 0$, hence, $B$ has a local maximum at $\theta_C$. 

7. **Sanity Check.**

\[
F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)} = \frac{\mu W}{B(\theta)}, \quad 0 \leq \theta \leq \pi/2
\]

\[F_{\text{min}} \text{ when } \theta = \theta_C = \arctan(\mu)\]

Pick a friendly value for \(\mu\), say \(\mu = 1\), so \(\theta_C = \pi/4\). Then:

\[
B(\theta) = \sin(\theta) + \cos(\theta) = \sqrt{2} \cos(\theta - \frac{\pi}{4}), \quad 0 \leq \theta \leq \pi/2
\]

Clearly, this is a cosine function shifted to the right by \(\pi/4\) radians and has a maximum at \(\theta = \pi/4\), which is consistent with our conclusion.

8. **Conclusion.**

\[F_{\text{min}} \text{ when } \theta = \theta_C = \arctan(\mu)\]
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. Present the analysis to find the dimensions of the field that has the largest area?

Let:

\[ A = \text{area enclosed by the fence in } \text{ft}^2 \]

\[ y = \text{length on fencing parallel to the river in } \text{ft} \]

\[ x = \text{length on fencing perpendicular to the river in } \text{ft} \]

\[ y \sim \text{ft} \]

\[ x \sim \text{ft} \]

\[ A = \text{area to be fenced } \sim \text{ft}^2. \]
§4.6 Optimisation

Applied Extrema – Example 2.2

\[ A = 2x(1200 - x) \]
\[ = 2(1200x - x^2) \]
\[ \frac{dA}{dx} = 0 = 2(1200 - 2x) \]
\[ = 4(600 - x) \implies x_c = 600 \]
\[ \frac{d^2A}{dx^2} = 4(0 - 1) < 0 \]

\[ A \text{ is maximum when } 2x + 600 = \frac{0}{4} \implies 1200 \]
\[ x = 2(1200 - x) \]
\[ y = 2400 - 2x \]
\[ = 2(1200 - x) \]
A closed cylindrical container is made to hold 1 litre of oil. Find the dimensions of container that will minimise the cost of the sheet metal to manufacture this container.

Let: $M = \text{sheetmetal cost per area in } \$/\text{cm}^2$

$L = \text{labour cost in }$

Then: $C = \text{total cost}$

$$C = M (2 \cdot \pi r^2) + M (2 \pi r \cdot h)$$

$$= M 2 \pi (r^2 + rh)$$

$$= M 2 \pi (r^2 + \frac{V}{\pi r^2})$$

$$\frac{dC}{dr} = M 2 \pi \left( 2r + \frac{V}{m} \left( -\frac{1}{r^2} \right) \right)$$

$$V = 1 L$$

$$V = \pi r^2 h$$

$$h = \frac{V}{\pi r^2}$$

$$= \frac{V}{\pi} r^{-2}$$
\[
\frac{dc}{dr} = 0 = 2\pi M \left( 2r - \frac{\sqrt{\frac{4}{\pi}}}{r^2} \right)
\]

\[
0 = 2r - \frac{\sqrt{\frac{4}{\pi}}}{r^2}
\]

\[
\frac{\sqrt{\frac{4}{\pi}}}{r^2} = 2r
\]

\[
\frac{\sqrt{\frac{4}{\pi}}}{\frac{1}{2}} = r^3 \implies r = \left(\frac{\sqrt{\frac{4}{\pi}}}{2}\right)^{\frac{1}{3}}
\]

\[
h = \frac{\sqrt{\frac{4}{\pi}}}{r^2} = \left(\frac{4V}{\pi}\right)^{\frac{1}{3}} = 2r
\]
§4.6 Optimisation

Applied Extrema – Example 3 sht 3
The quantity of soot deposited on the ground by a smokestack is inversely proportional to the square of the distance from the stack. Two smokestacks are 20 miles apart. The quantity of soot on the line joining them is the sum of the soot from each stack. One of the smokestacks generates eight times the quantity of soot of the other. Present the analysis to find the point on the line joining the stacks where the soot is a minimum.
§4.6 Optimisation

Applied Extrema – A Example 4 (anal sht 1)

\[ S_1(x) = \frac{k_1 s_0}{x^2}, \quad S_2(x) = \frac{k_2 s_0}{(20-x)^2} \]

\[ S(x) = S_1(x) + S_2(x) = \frac{k_1 s_0}{x^2} + \frac{k_2 s_0}{(20-x)^2} \]
\[ S(x) = \frac{k_1 s_0}{x^2} + \frac{8k_1 s_0}{(20-x)^2} \]

\[ = k_1 s_0 \left( \frac{1}{x^2} + \frac{8}{(20-x)^2} \right) \]

\[ \frac{dS}{dx} = k_1 s_0 \left( -\frac{2}{x^3} + \frac{8 \cdot 2}{(20-x)^3} \right) \]

\[ 0 = -\frac{1}{x^3} + \frac{8}{(20-x)^3} \]

\[ \frac{1}{x^3} = \frac{8}{(20-x)^3} \]

\[ \frac{(20-x)^3}{x^3} = 8 \Rightarrow \frac{20-x}{x} = 2 \Rightarrow \frac{20}{x} - 1 = 2 \Rightarrow \frac{20}{x} = 3 \]
\[
\frac{x}{20} = \frac{1}{3} \Rightarrow x = \frac{20}{3}
\]

\[
\therefore S_{\text{min}} = S\left(\frac{20}{3}\right)
\]
Let $Q(x) = \text{quantity of soot from both smokestacks, } 0 < x < 20$. Then:

$$Q(x) = \frac{k}{x^2} + \frac{8k}{(20-x)^2}, k \in \mathbb{R} \land k \geq 0$$

$$q(x) = \frac{Q(x)}{k} = \frac{1}{x^2} + \frac{8}{(20-x)^2}$$

Caveat! A calculator might rewrite this as:

$$q(x) = \frac{Q(x)}{k} = \frac{1}{x^2} + \frac{8}{(x-20)^2}$$
§4.6 Optimisation
Applied Extrema – A Example 4 (anal sht 3)

\[ q(x) = \frac{1}{x^2} + \frac{8}{(20-x)^2} \]
§4.6 Optimisation

Applied Extrema – Snell's Law from Chemistry & Physics

Let \( v_1 \) be the velocity of light in air and \( v_2 \) be the velocity of light in water. According to Fermat's Principle, a ray of light will travel from point \( A \) in air to point \( B \) in the water by path \( ACB \) that minimises the time taken. Show that this is consistent with Snell's Law:

\[
\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}
\]

where the incident angle \( \theta_1 \) and refraction angle \( \theta_2 \) are as shown.

Let the photon travel time be:

\[
T(x) = (\text{time } A \text{ to } C) + (\text{time } C \text{ to } B)
\]

\[
T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}
\]

\[
\frac{dT}{dx} = \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{d-x}{v_2 \sqrt{b^2 + (d-x)^2}}
\]