The Mean Value Theorem If $f$ is a differentiable function on the interval $[a,b]$, then there exists a number $c$ between $a$ and $b$ such that

1. $f'(c) = \frac{f(b) - f(a)}{b - a}$

or equivalently,

2. $f(b) - f(a) = f'(c)(b - a)$

$m_{AB} = \frac{f(b) - f(a)}{b - a}$

$f'(c_1) = \frac{f(b) - f(a)}{b - a}$

$f'(c_2) = \frac{f(b) - f(a)}{b - a}$
Increasing/Decreasing Test

(a) If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval.

(b) If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on that interval.

Proof

Let \( x_1 \) and \( x_2 \) be two numbers on a specified interval and \( x_1 < x_2 \). If \( f \) is differentiable on \([x_1, x_2]\) then by the Mean Value Theorem:

\[
f(x_2) - f(x_1) = f'(c) (x_2 - x_1)
\]

Here \( x_2 - x_1 > 0 \). Therefore:

- if \( f'(x) > 0 \) \( \Rightarrow \) \( f(x_2) - f(x_1) > 0 \) and \( f \) is increasing,
- if \( f'(x) < 0 \) \( \Rightarrow \) \( f(x_2) - f(x_1) < 0 \) and \( f \) is decreasing.
§4.3 Derivatives & the Shapes of Curves (MTH_251 Review)
Increasing & Decreasing Test - Example

Locate the intervals on which the function \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \) is increasing and where it is decreasing.

**Solution:**

\[
f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)
\]

By the I/D Test \( f(x) \) is increasing when \( f'(x) > 0 \) and decreasing when \( f'(x) < 0 \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>12x</th>
<th>( x - 2 )</th>
<th>( x + 1 )</th>
<th>( f'(x) )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -1 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>decreasing on ((-\infty, -1))</td>
</tr>
<tr>
<td>(-1 &lt; x &lt; 0 )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>increasing on ((-1, 0))</td>
</tr>
<tr>
<td>( 0 &lt; x &lt; 2 )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>decreasing on ((0, 2))</td>
</tr>
<tr>
<td>( x &gt; 2 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing on ((2, \infty))</td>
</tr>
</tbody>
</table>
The First Derivative Test

Suppose that $c$ is a critical number of a continuous function $f$.

(a) If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$.

(b) If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$.

(c) If $f'$ does not change sign at $c$, then $f$ has no local extremum at $c$. 
§4.3 Derivatives & the Shapes of Curves (MTH_251 Review)
The First Derivative Test  Example

Examining the graph of \( f(x) \), we see that:

- \( c_1 \) is a critical number but there no local extremum at \( c_1 \) because \( f' > 0 \) to the left and to the right of \( c_1 \).
- \( c_2 \) is a critical number and there a local maximum at \( c_2 \) because \( f' > 0 \) to the left and \( f' < 0 \) to the right of \( c_2 \).
- \( c_3 \) is a critical number and there no local extremum at \( c_3 \) because \( f' < 0 \) to the left and to the right of \( c_3 \).
§4.3 Derivatives & the Shapes of Curves (MTH_251 Review)
Concavity and the Concavity Test

From §2.10, a function (or its graph) is called *concave upward* on an interval $I$ if $f'$, the rate of change of $f$, is an increasing function on $I$. It is called *concave downward* on $I$ if $f'$ is decreasing on $I$.

The point at which a function changes concavity is called an *inflection point*.

Concavity is determined by the change in $f'$, hence related to $f''$.

**Concavity Test**
(a) If $f''(x) > 0$ for all $x$ in $I$, the graph of $f$ is concave upward on $I$.
(b) If $f''(x) < 0$ for all $x$ in $I$, the graph of $f$ is concave downward on $I$. 
§4.3 Derivatives & the Shapes of Curves (MTH_251 Review)

The Second Derivative Test

From the concavity test, there is a point of inflection wherever the second derivative changes sign. A test for extrema is then:

The Second Derivative Test

Suppose that $f''$ is continuous near $c$.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

Restated we could say that if $f''$ is continuous near $c$ and

(a) if $c$ is a critical number and $f$ is concave upward (i.e., $f''(x) > 0$), then $f$ has a local minimum at $c$,
(b) if $c$ is a critical number and $f$ is concave downward, then $f$ has a local maximum at $c$. 
§4.3 Derivatives & the Shapes of Curves (MTH_251 Review)

The Behaviour of a Function

The Kling recipe for analysing the behaviour of function $f(x)$.

1. Compute the first and second derivatives of $f$.
2. Determine the critical numbers of $f$, i.e., the numbers in the domain of $f$ such that either $f'(c) = 0$ or $f''(c)$ does not exist.
3. Make a table with columns containing the intervals specified by the critical numbers, the sign of $f'$, and a description of the implied behaviour of the function on the interval.
4. Find the possible inflection points of $f$.
5. Make a table with columns containing the intervals specified by the possible inflection points and discontinuities, the sign of $f''$, and a description of the implied behaviour of the function on the interval.
6. Plot the extreme points, inflection points, asymptotes, and any friendly points.
7. Connect the dots appropriately.
The figure show a beam of length $L$ embedded in concrete walls. If a constant load $W$ is distributed evenly along its length, the beam takes the shape of a deflection curve

$$y = -\frac{W}{24EI} x^4 + \frac{WL}{12EI} x^3 - \frac{WL^2}{24EI} x^2$$

where $E$ and $I$ are positive constants. ($E$ is Young's modulus of elasticity and $I$ is the moment of inertia of the cross section of the beam.) Present the analysis to determine the characteristics of the deflection curve.
\[ y = -\frac{W}{24EI} x^4 + \frac{WL}{12EI} x^3 - \frac{WL^2}{24EI} x^2 \]

\[ = -\frac{W}{24EI} \left( x^4 - 2xL + L^2 \right) \]

\[ = -\frac{W}{24EI} \left( \left( \frac{x}{L} \right)^4 - 2 \left( \frac{x}{L} \right)^3 + \left( \frac{x}{L} \right)^2 \right) \]

\[ = -\frac{WL^4}{24EI} \left( 5 - 2\xi^2 + \xi^4 \right) \]

\[ y(\xi(x)) = \frac{dy}{dx} = 0 = -\frac{WL^4}{24EI} \left( 4\xi^3 \frac{d\xi}{dx} - 2\cdot3\xi^2 \frac{d\xi}{dx} + 2\xi \frac{d^2\xi}{dx^2} \right) \]