§4.2 Maximum and Minimum Values (MTH_251 Review)

Derivatives

Let $f(x)$ be a continuous function on a specified interval.

Let $f' = df/dx =$ derivative of $f$:
- If $f' > 0 \Rightarrow f$ is increasing
- If $f' < 0 \Rightarrow f$ is decreasing
- If $f' = 0 \Rightarrow f$ is stationary on the specified interval.

Let $f'' = d^2f/dx^2 =$ derivative of $f'$:
- If $f'' > 0 \Rightarrow f'$ is increasing
- If $f'' < 0 \Rightarrow f'$ is decreasing
- If $f'' = 0 \Rightarrow f'$ is stationary on the specified interval.
Let $f(x)$ be a continuous function on a specified interval.

Let $f' = df/dx = \text{derivative of } f$:
- If $f' > 0 \Rightarrow f$ is increasing
- If $f' < 0 \Rightarrow f$ is decreasing
- If $f' = 0 \Rightarrow f$ is stationary on the specified interval.

Let $f'' = d^2f/dx^2 = \text{derivative of } f'$:
- If $f'' > 0 \Rightarrow f'$ is increasing
- If $f'' < 0 \Rightarrow f'$ is decreasing
- If $f'' = 0 \Rightarrow f'$ is stationary on the specified interval.
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More On Derivatives

- If $f'' > 0$ on an interval $\Rightarrow f''$ is increasing over that interval $\Rightarrow$ graph of $f$ is concave up on that interval
- If $f'' < 0$ on an interval $\Rightarrow f'$ is decreasing over that interval $\Rightarrow$ graph of $f$ is concave down on that interval
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Absolute (Global) Extrema

1. Definition Let \( c \) be a number in the domain \( D \) of the function \( f \). Then, \( f(c) \) is the
   • absolute maximum value of \( f \) if \( f(c) \geq f(x) \ \forall x \in D \).
   • absolute minimum value of \( f \) if \( f(c) \leq f(x) \ \forall x \in D \).
1. **Definition** Let \( c \) be a number in the domain \( D \) of the function \( f \). Then, \( f(c) \) is the
   - absolute maximum value of \( f \) if \( f(c) \geq f(x) \) \( \forall x \in D \).
   - absolute minimum value of \( f \) if \( f(c) \leq f(x) \) \( \forall x \in D \).

2. **Definition** The number \( f(c) \) is a
   2. local maximum value of \( f \) if \( f(c) \geq f(x) \) when \( x \) is near \( c \).
   3. local minimum value of \( f \) is \( f(c) \leq f(x) \) when \( x \) is near \( c \).
Identifying & Classifying Extrema

Given the graph of $f(x)$, state whether $f(x)$ has an absolute max/min, a local max/min or neither a max nor min at $x_1, x_2, etc.$

<table>
<thead>
<tr>
<th>Point</th>
<th>Abs Max</th>
<th>Abs Min</th>
<th>Local Max</th>
<th>Local Min</th>
<th>No Extrema</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
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<tr>
<td>$x_3$</td>
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<td>$x_4$</td>
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</tr>
<tr>
<td>$x_5$</td>
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</tr>
</tbody>
</table>
3. **The Extreme Value Theorem** If \( f \) is a continuous function on a closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\).

**Corollary** If \( f \) is NOT a continuous function on a closed interval \([a, b]\), then \( f \) need not attain an absolute extrema on \([a, b]\).
Fermat's Theorem

4. **Fermat's Theorem**  If $f$ has a local extremum at $c$, and if $f'(c)$ exists, then $f''(c) = 0$.  

**N.B.:** Fermat's Theorem does not state that $f'(c) = 0$ means that $f(c)$ is an extremum.

**e.g.,** If $f(x) = x^3$, $f'(0) = 0$. However, $f(x)$ does NOT have an extremum at $x = 0$. 
§4.2 Maximum and Minimum Values (MTH_251 Review)

Critical Numbers

5. **Definition** A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f''(c)$ does not exist.

6. **Fermat's Theorem Restated:** If $f$ has a local extremum at $c$, then $c$ is a critical number of $f$.

\[ f(x) = (x - 1)^3 \]
\[ f'(x) = 3(x - 1)^2 \]

$f'(1) = 0$, hence $x = 1$ is a critical number. However $f(1)$ is NOT an extremum.

\[ f(x) = |x - 1| \]
\[ f'(x) = \begin{cases} -1, & x < 1 \\ 1, & x > 1 \end{cases} \]
The Closed Interval Method – Absolute Extrema On a Closed Interval

The Closed Interval Method: To find absolute extrema values of a continuous function $f$ on a closed interval $[a,b]$.

1. Find the values of $f$ at the critical numbers of $f$ in $(a,b)$.
2. Find the values of $f$ at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum and the smallest of the values from Steps 1 and 2 is the absolute minimum.