§5.2 Trigonometric Functions: Unit Circle Approach

Objectives

1. Find the Exact Values of the Trigonometric Functions Using a Point on the Unit Circle
2. Find the Exact Values of the Trigonometric Functions of Quadrantal Angles
3. Find the Exact Values of the Trigonometric Functions of $\pi/4 = 45^\circ$
4. Find the Exact Values of the Trigonometric Functions of $\pi/6 = 30^\circ$ and $\pi/3 = 60^\circ$
5. Find the Exact Values of the Trigonometric Functions for integer multiples of $\pi/6 = 30^\circ$, $\pi/4 = 45^\circ$, and $\pi/3 = 60^\circ$
6. Use a Calculator to Approximate the Values of a Trigonometric Function
7. Use a Circle of Radius $r$ to Evaluate the Trigonometric Functions
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The Length of an Arc of a Circle

Angle in radians:

\[ \theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} \]

1 revolution \( \frac{2\pi r}{r} = 2\pi \)

N.B.:
- An angle in radians is length over length, hence a dimensionless quantity.
- For a circle of radius \( r \), a central angle \( \theta \) in radians subtends an arc of length \( s \) such that:
  \[ s = r \theta \]
§5.2 Trigonometric Functions: Unit Circle Approach

Unit Circle and a Point on a Unit Circle

A unit circle is a circle of radius 1 unit centered at the origin of a Cartesian (i.e., rectangular) coordinate system.

Let $t$ be a real number and let $P = (x, y)$ be the point on the unit circle that corresponds to $t$. 

Let $s = t > 0$ then $P = (x, y)$ will be in the first quadrant.

Let $s = t < 0$ then $P = (x, y)$ will be in the fourth quadrant.
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Trigonometric Functions on a Unit Circle

Let $t$ be a real number and let $P = (x, y)$ be the point on the unit circle (i.e., a circle with $r = 1$) that corresponds to $t$.

The **sine function** associates with $t$ the $y$-coordinate of $P$ and is denoted by:

$$\sin(t) = \frac{y}{r} = \frac{y}{1}$$

The **cosine function** associates with $t$ the $x$-coordinate of $P$ and is denoted by:

$$\cos(t) = \frac{x}{r} = \frac{x}{1}$$

If $x \neq 0$, the **tangent function** associates with $t$ the ratio of the $y$-coordinate to the $y$-coordinate of $P$ and is denoted by:

$$\tan(t) = \frac{y}{x}$$

If $y \neq 0$, the **cosecant function** defined as:

$$\csc(t) = \frac{r}{y} = \frac{1}{y}$$

If $x \neq 0$, the **secant function** defined as:

$$\sec(t) = \frac{r}{x} = \frac{1}{x}$$

If $y \neq 0$, the **cotangent function** defined as:

$$\cot(t) = \frac{x}{y}$$
Let $t$ be a real number and let $P = \left( -\frac{1}{5}, \frac{2\sqrt{6}}{5} \right)$ be a point on the unit circle that corresponds to $t$.

Find the values of $\sin(t)$, $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, and $\cot(t)$.

\begin{align*}
\sin(t) &= \frac{y}{r} = \frac{2\sqrt{6}}{5} \\
\cos(t) &= \frac{x}{r} = -\frac{1}{5} \\
\tan(t) &= \frac{y}{x} = \left( \frac{2\sqrt{6}}{5} \right) \left( -\frac{5}{1} \right) = -2\sqrt{6} \\
\csc(t) &= \left( \frac{r}{y} \right) = \frac{5}{2\sqrt{6}} \\
\sec(t) &= \left( \frac{r}{x} \right) = -5 \\
\cot(t) &= \left( \frac{x}{y} \right) = -\frac{1}{2\sqrt{6}}
\end{align*}
§5.2 Trigonometric Functions: Unit Circle Approach
Trigonometric Functions of Angles

\[
\begin{align*}
\sin(\theta) &= \frac{y}{r} = \frac{\theta_1}{r} \\
\cos(\theta) &= \frac{x}{r} = \frac{-x}{r} \\
\sin(\theta_2) &\neq 0 \\
\cos(\theta_2) &\neq 0 \\
\tan(\theta_1) &> 0 \\
\tan(\theta_2) &< 0 \\
\tan(\theta_3) &> 0 \\
\tan(\theta_4) &< 0 \\
\end{align*}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

Trigonometric Functions of Angles - Definition

If $\theta = t$ radians, the **six trigonometric functions of the angle $\theta$** are defined as:

\[
\begin{align*}
\sin(t) &= \sin(\theta), & \cos(t) &= \cos(\theta), & \tan(t) &= \tan(\theta) \\
\csc(t) &= \csc(\theta), & \sec(t) &= \sec(\theta), & \cot(t) &= \cot(\theta)
\end{align*}
\]
2. **Exact Values of Trigonometric Functions of Quadrantal Angles**

Find exact values of the six trigonometric functions of:

a) $\theta = 0 = 0^\circ$

- $\sin(0) = 0$
- $\cos(0) = 1$
- $\tan(0) = 0$
- $\csc(0)$ is undefined
- $\sec(0)$ is undefined
- $\cot(0)$ is undefined

b) $\theta = \pi/2 = 90^\circ$

- $\sin(90^\circ) = 1$
- $\cos(90^\circ) = 0$
- $\tan(90^\circ)$ is undefined
- $\csc(90^\circ) = 1$
- $\sec(90^\circ)$ is undefined
- $\cot(90^\circ) = 0$
§5.2 Trigonometric Functions: Unit Circle Approach

2. Exact Values of Trigonometric Functions of Quadrantal Angles

Find exact values of the six trigonometric functions of:

a) \( \theta = \pi = 180^\circ \)

\[ \sin(\pi) = 0 \]
\[ \cos(\pi) = -1 \]
\[ \tan(\pi) = 0 \]

b) \( \theta = 3\pi/2 = 270^\circ \)

\[ \sin(3\pi/2) = -1 \]
\[ \cos(3\pi/2) = 0 \]
\[ \tan(3\pi/2) \text{ undefined} \]
§5.2 Trigonometric Functions: Unit Circle Approach

Exact Values of Trigonometric Functions of Quadrantal Angles

<table>
<thead>
<tr>
<th>Quadrantal Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ (rad)</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>π/2</td>
</tr>
<tr>
<td>π</td>
</tr>
<tr>
<td>3π/2</td>
</tr>
</tbody>
</table>

und = undefined
Find the exact value of:

(a) \( \sin(5\pi) = \sin(2\pi + 3\pi) = 0 \)

(b) \( \cos(-540°) = \cos(-360° - 180°) = -1 \)
§5.2 Trigonometric Functions: Unit Circle Approach

3. Exact Values of Trigonometric Functions of $\theta = \pi/4$

\[
x^2 + y^2 = r^2
\]
\[
x = y
\]
\[
2x^2 = r^2
\]
\[
x^2 = \frac{1}{2} r^2
\]
\[
x = \frac{r}{\sqrt{2}}
\]
\[
\cos\left(\frac{\pi}{4}\right) = \frac{x}{r} = \frac{1}{\sqrt{2}} \cdot \frac{1}{r}
\]
\[
\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}
\]
\[
\tan\left(\frac{\pi}{4}\right) = 1
\]
§5.2 Trigonometric Functions: Unit Circle Approach

4. Exact Values of Trigonometric Functions of $\theta = \pi/3$ & $\pi/6$

\[
\begin{align*}
\cos \left( \frac{\pi}{3} \right) &= \frac{1}{2}, & \sin \left( \frac{\pi}{3} \right) &= \frac{\sqrt{3}}{2} \\
x^2 + y^2 &= 2^2 \\
1^2 + y^2 &= 2^2 \\
y^2 &= 3 \\
y &= \sqrt{3} \\
\sin \left( \frac{\pi}{3} \right) &= \frac{\sqrt{3}}{2} \\
\tan \left( \frac{\pi}{3} \right) &= \frac{\sqrt{3}}{2} \\
\left( \frac{1}{2} \right)^2 + y^2 &= 1^2 \\
y^2 &= 3/4 \\
y &= \frac{\sqrt{3}}{2}
\end{align*}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

4. Exact Values of Trigonometric Functions of $\theta = \pi/3 \& \pi/6$

$x = r/2$

$x^2 + y^2 = r^2$

$(r/2)^2 + y^2 = r^2$

$y^2 = r^2 - \frac{1}{4}r^2$

$y^2 = \frac{3}{4}r^2$

$y = \pm \frac{\sqrt{3}}{2}r$

$\sin \left( \frac{\pi}{3} \right) = \frac{y}{r} = \frac{\sqrt{3}}{2}$

$\cos \left( \frac{\pi}{3} \right) = \frac{x}{r} = \frac{1}{2}$
§5.2 Trigonometric Functions: Unit Circle Approach

4. Exact Values of Trigonometric Functions of $\theta = \pi/3$ & $\pi/6$

\[
\begin{align*}
\sin \left( \frac{\pi}{6} \right) &= \cos \left( \frac{\pi}{3} \right) = \frac{y}{r} = \frac{1}{2} \\
\cos \left( \frac{\pi}{6} \right) &= \sin \left( \frac{\pi}{3} \right) = \frac{x}{r} = \frac{\sqrt{3}}{2}
\end{align*}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

4. Exact Values of a Trigonometric Expression

Find the exact values of each expression.

(a) \( z_1 = \sin(180^\circ) \cos(45^\circ) \)

\[
= 0
\]

(b) \( z_2 = \sin\left(\frac{\pi}{4}\right) \cos(\pi) - \sin\left(\frac{3\pi}{2}\right) \)

\[
= \frac{\sqrt{2}}{2}(-1) - (-1)
\]

(c) \( z_3 = \left(\csc\left(\frac{\pi}{4}\right)\right)^2 + \tan\left(\frac{\pi}{4}\right) \)

\[
= (\sqrt{2})^2 + 1
\]

\[
= 3
\]
§5.2 Trigonometric Functions: Unit Circle Approach
Exact Values of Trigonometric Functions

<table>
<thead>
<tr>
<th>θ (rad)</th>
<th>θ (°)</th>
<th>sin(θ)</th>
<th>cos(θ)</th>
<th>tan(θ)</th>
<th>csc(θ)</th>
<th>sec(θ)</th>
<th>cot(θ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>π/6</td>
<td>30</td>
<td>1/2</td>
<td>(\sqrt{3}/2)</td>
<td>1/√3</td>
<td>2</td>
<td>2/√3</td>
<td>√3</td>
</tr>
<tr>
<td>π/4</td>
<td>45</td>
<td>(\sqrt{2}/2)</td>
<td>(\sqrt{2}/2)</td>
<td>1</td>
<td>√2</td>
<td>√2</td>
<td>1</td>
</tr>
<tr>
<td>π/3</td>
<td>60</td>
<td>(\sqrt{3}/2)</td>
<td>1/2</td>
<td>√3</td>
<td>2/√3</td>
<td>2</td>
<td>1/√3</td>
</tr>
</tbody>
</table>
The path of a projectile with initial velocity $v_0$ and trajectory angle $\theta$ is a parabola. The horizontal distance travelled by the projectile is its range $R$:

$$R(\theta) = \frac{v_0^2 \sin(2\theta)}{g}$$

Where $g$ is the acceleration due to gravity. The maximum height $H$ of the projectile is:

$$H(\theta) = \frac{v_0^2 \sin^2(\theta)}{2g}$$

Find $R$ and $H$ if $v_0 = 150 \text{ m/s}$, $\theta = 30^\circ$, with $g = 9.8 \text{ m/s}^2$. 

\[ g = 3.2 \text{ ft/s}^2 \]
A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at angle $\theta$ as shown in the figure. The area $A$ of the opening may be expressed as a function $\theta$ as:

$$A(\theta) = 16 \sin(\theta) (\cos(\theta) + 1)$$

Find the area $A$ of the opening for $\theta = 30^\circ$, $\theta = 45^\circ$, and $\theta = 60^\circ$
### Example Application – Constructing a Rain Gutter

\[ A(\theta) = 16 \sin(\theta) (\cos(\theta) + 1) \]

<table>
<thead>
<tr>
<th>( \theta (\degree) )</th>
<th>( A(\theta) ) ((\text{in}^2))</th>
<th>Approx ( A(\theta) ) ((\text{in}^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
§5.2 Trigonometric Functions: Unit Circle Approach

5. **Exact** Values of Trig Functions for Integer Multiples of $\pi/6$, $\pi/4$ & $\pi/3$

\[
x = y
\]
\[
x^2 + y^2 = r^2
\]
\[
x^2 + x^2 = r^2
\]
\[
2x^2 = r^2
\]
\[
x^2 = r^2 / 2
\]
\[
x = y = \pm \frac{r}{\sqrt{2}}
\]

\[
\sin(\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\cos(\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

5. **Exact** Values of Trig Functions for Integer Multiples of $\pi/6$, $\pi/4$ & $\pi/3$

\[
\sin(\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]

\[
\cos(\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

5. **Exact** Values of Trig Functions for Integer Multiples of $\pi/6$, $\pi/4$ & $\pi/3$

\[
\sin(\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\cos(\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\sin(3\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\cos(3\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

5. **Exact** Values of Trig Functions for Integer Multiples of $\pi/6$, $\pi/4$ & $\pi/3$

\[
\begin{align*}
\sin(\pi/4) &= \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\cos(\pi/4) &= \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\sin(3\pi/4) &= \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\cos(3\pi/4) &= \frac{x}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\
\sin(5\pi/4) &= \frac{y}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\
\cos(5\pi/4) &= \frac{x}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\end{align*}
\]
§5.2 Trigonometric Functions: Unit Circle Approach

5. **Exact** Values of Trig Functions for Integer Multiples of $\pi/6$, $\pi/4$ & $\pi/3$

\[
\sin(\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\cos(\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\sin(3\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]
\[
\cos(3\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\]
\[
\sin(5\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\]
\[
\cos(5\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\]
\[
\sin(-\pi/4) = \frac{y}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\]
\[
\cos(-\pi/4) = \frac{x}{r} = \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}
\]
Find the exact values of:

(a) \( \cos(135^\circ) = -\frac{\sqrt{2}}{2} \)

(b) \( \tan\left(-\frac{\pi}{4}\right) = -1 \)

(c) \( \sin(315^\circ) = \)
Find the exact values of:

(a) \( \tan \left( \frac{7\pi}{6} \right) = \)

(b) \( \sin(-120^\circ) = \)

(c) \( \cos \left( -\frac{2\pi}{3} \right) = -\frac{1}{2} \)

(d) \( \sin(\theta) = \frac{1}{2} \Rightarrow \sin(\theta + \pi) = -\frac{1}{2} \)

\[ \sin(\theta + \pi) = -\sin(\theta) + \sin(\pi) = \frac{1}{2} + 0 = \frac{1}{2} \]
Use a calculator to approximate the value of the trigonometric function rounded to two decimal places:

(a) \( \cos(48^\circ) \approx 0.67 \)

(b) \( \tan\left(\frac{\pi}{12}\right) \approx 0.27 \)

(c) \( \sec(10) \approx \frac{1}{\cos(10)} \approx -1.19 \)
For an angle \( \theta \) in standard position, let \( P = (x, y) \) be the point on the terminal side of \( \theta \) that is also on the circle \( x^2 + y^2 = r^2 \). Then

\[
\begin{align*}
\sin(\theta) &= \frac{y}{r} \\
\cos(\theta) &= \frac{x}{r} \\
\tan(\theta) &= \frac{y}{x}, \quad x \neq 0 \\
\csc(\theta) &= \frac{r}{y}, \quad y \neq 0 \\
\sec(\theta) &= \frac{r}{x}, \quad x \neq 0 \\
\cot(\theta) &= \frac{x}{y}, \quad y \neq 0
\end{align*}
\]
Find the **exact values** of each of the six trigonometric functions of an angle $\theta$ if $(-3, 3)$ is a point on its terminal side.

\[ x^2 + y^2 = r^2 \]

\[ r^2 = (-3)^2 + 3^2 \]

\[ = 18 \]

\[ r = \sqrt{18} = 3\sqrt{2} \]

\[ \sin (\theta) = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \]

\[ \cos (\theta) = \frac{-3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}} \]

\[ \tan (\theta) = \frac{-3}{-3} = 1 \]

\[ \csc (\theta) = \sqrt{2} \]

\[ \sec (\theta) = -\sqrt{2} \]

\[ \cot (\theta) = -1 \]