Section 8.8: The Binomial Series

Text begins with a review of what is called the Binomial Theorem which appears in some algebra courses.

Let $k$ be a positive integer, then:

$$(a + b)^k = a^k + ka^{k-1}b + \frac{k(k+1)}{2!}a^{k-2}b^2 + \frac{k(k+1)(k+2)}{3!}a^{k-3}b^3 + \ldots + \frac{k(k+1)(k+2)\ldots(k+n-1)}{n!}a^{k-n}b^n + \ldots + k^n b^k$$

There is a simplified notation which is used to indicate the product in the general term:

$$\binom{k}{0} = 1 \quad \binom{k}{1} = \frac{k(k+1)(k+2)\ldots(k+n-1)}{n!} \quad n = 1, 2, \ldots, k$$

With this notation, the binomial theorem can be written as:

$$(a + b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}b^n$$

As a special case, if we let $a = 1$ and $b = x$, we get

$$(1 + x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

Consider the function $f(x) = (1 + x)^k$ and derive the Maclaurin series.

$$f(x) = (1 + x)^k \quad f(0) = 1$$
$$f'(x) = k(1 + x)^{k-1} \quad f'(0) = k$$
$$f''(x) = k(k+1)(1 + x)^{k-2} \quad f''(0) = k(k+1)$$
$$f'''(x) = k(k+1)(k+2)(1 + x)^{k-3} \quad f'''(0) = k(k+1)(k+2)$$

... 

$$f^{(n)}(x) = k(k+1)(k+2)\ldots(k+n-1)(1 + x)^{k-n} \quad f^{(n)}(0) = k(k+1)(k+2)\ldots(k+n-1)$$

$$\sum_{n=0}^{k} \binom{k}{n} (1 + x)^{k-n} = \sum_{n=0}^{k} \binom{k}{n} x^n$$

This formula is what is known as the Binomial Series.

If we use the Ratio test on this series, we have:
The Binomial Series: If \( k \) is any real number and \(|x| < 1\), then

\[
(1 + x)^k = 1 + kx + \frac{k(k+1)}{2!}x^2 + \frac{k(k+1)(k+2)}{3!}x^3 + \ldots
\]

where \( \begin{pmatrix} k \\ n \end{pmatrix} = \frac{k(k+1) \ldots (k+n)}{n!} \) (for \( n = 0 \) and \( n = 1 \))

As the text points out without any derivation, this series converges for \( x = 1 \) if -1 < \( k \leq 0 \) and at both endpoints if \( k > 0 \).

Also note that if \( k \) is a positive integer and \( n > k \), then the expression \( \begin{pmatrix} k \\ n \end{pmatrix} \) contains a factor of \((k-k)\) and so is zero. Therefore the series is a finite series and reduces to the ordinary Binomial Theorem.

Ex's of using the Binomial Series:

Find a power series and the radius of convergence for the following:

\[
\sqrt{1 + x} = (1 + x)^{1/2} = \sum_{n=0}^{\infty} \begin{pmatrix} 1/2 \\ n \end{pmatrix} x^n = 1 + \frac{1}{2}x + \frac{1}{2 \cdot 2!}x^2 + \frac{1}{2 \cdot 2 \cdot 3!}x^3 + \ldots
\]

\[
= 1 + \frac{x}{2} + \frac{3x^3}{2^2 \cdot 2!} + \frac{3 \cdot 5 x^4}{2^3 \cdot 3!} + \ldots
\]

\[
= 1 + \frac{x}{2} + \sum_{n=0}^{\infty} \begin{pmatrix} 1/2 \\ n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3) x^n
\]

\[
R = 1, \quad I = [-1, 1] \quad \text{(not derived, used text info on convergence)}
\]
Now let \( x = \frac{1}{2} \). To get:

\[
\sum_{n=1}^{\infty} \frac{n(n+1)}{2^n} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^n} = 6
\]