Sample Problems # 2

1. For the following integral the technique of u-substitution may be used to find the definite integral.

\[ \int_{6}^{8} (x - 3) \sqrt{x^2 - 6x} \, dx \]

State the choice of u which will allow u-substitution to perform the integral.

\[ u = x^2 - 6x \quad \Rightarrow \quad du = (2x - 6) \, dx \]

\[ \frac{1}{2} \, du = (x - 3) \, dx \]

Apply the method of u-substitution to get the integral.

\[ \int_{6}^{8} (x - 3) \sqrt{x^2 - 6x} \, dx = \int_{0}^{16} \frac{1}{2} (u)^{\frac{1}{2}} \, du = \frac{1}{2} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_{0}^{16} \]

\[ = \frac{1}{3} \left[ 16^{\frac{3}{2}} - 0^{\frac{3}{2}} \right] = \frac{1}{3} \left[ 64 - 0 \right] = \frac{64}{3} \]
2. For the following integral, Integration By Parts may be used to find the **indefinite integral**.

\[ \int x^3 e^{x^2} \, dx \]

From the integral form \( \int u \, dv \), state the choices for \( u \) and \( dv \) which will allow Integration By Parts to perform the integral.

\[ u = x^2 \quad \quad \quad \quad \quad \quad dv = xe^{x^2} \, dx \]

State the resulting \( du \) and \( v \) from your choices above.

\[ du = 2x \, dx \quad \quad \quad \quad \quad \quad v = \frac{1}{2} e^{x^2} \]

\[ \int xe^{x^2} \, dx = \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C \]

Apply Integration By Parts to get the integral.

\[ \int x^3 e^{x^2} \, dx = \left[ x^2 \left( \frac{1}{2} \right) e^{x^2} \right] - \int 1x e^{x^2} \, dx \]

\[ = \frac{1}{2} x^2 e^{x^2} - \int \frac{1}{2} e^u \, du \]

\[ = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C \]
3. For the following integral, Integration By Parts may be used to find the definite integral.

\[
\int_{0}^{\pi/3} \frac{\sin(x) \ln(\cos x)}{\cos^2 x} \, dx
\]

From the integral form \( \int u \, dv \), state the choices for \( u \) and \( dv \) which will allow Integration By Parts to perform the integral.

\[
\begin{align*}
u &= \ln(\cos x) & \quad d\nu &= \sin(x)(\cos x)^{-2} \, dx \\
\end{align*}
\]

State the resulting \( du \) and \( v \) from your choices above.

\[
\begin{align*}
\frac{dv}{du} &= \cos(x) \\
\int \frac{\sin(x)(\cos x)^{-2}}{\cos^2 x} \, dx &= \int \cos(x) \, d\left(\cos(x)^{-1}\right)
\end{align*}
\]

Apply Integration By Parts to get the integral.

\[
\begin{align*}
\int_{0}^{\pi/3} \frac{\sin(x)(\cos x)}{\cos^2 x} \, dx &= \left[ \ln(\cos x) \sec x \right]_{0}^{\pi/3} - \int_{0}^{\pi/3} \sec x (-\tan x) \, dx \\
&= \ln(\cos(\pi/3)) \sec(\pi/3) - 0 + \left[ \sec x \right]_{0}^{\pi/3} \\
&= \ln(1/2) + \left( \sec(\pi) - \sec 0 \right) = 2 \ln(1/2) + 2 - 1 \\
&= 2 \ln(1/2) + 1
\end{align*}
\]
4. Determine if the Improper Integral converges or diverges. If it converges, find its value. Show your work.

\[ \int_{\frac{1}{2}}^{\infty} \frac{1}{x(\ln x)^2} \, dx \]

\[
\lim_{t \to \infty} \int_{\frac{1}{2}}^{t} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to \infty} \int_{\frac{1}{2}}^{t} \frac{1}{u^2} \, du \quad \text{with} \quad u = \ln x, \quad du = \frac{1}{x} \, dx
\]

\[
= \lim_{t \to \infty} \left[ \frac{-1}{(\ln t)^2} \right]_{\frac{1}{2}}^{t} = \lim_{t \to \infty} \left[ \frac{-1}{(\ln t)^2} + \frac{1}{(\ln \frac{1}{2})^2} \right]
\]

\[
= 0 + \frac{1}{(\ln \frac{1}{2})^2} = \frac{1}{(\ln 2)^2}
\]

Therefore, the improper integral converges and it has the value \(\frac{1}{(\ln 2)^2}\).
5. Compute the area of the finite region enclosed by the curves.

\[ y = |x| \quad \text{for } x > 0 \quad x = 6 - x^2 \]

\[ y = 6 - x^2 \quad x^2 + x - 6 = 0 \]

\[ (x+3)(x-2) = 0 \]

Find the points of intersection and integrate over the finite interval.

Using interval \(-2 \leq x \leq 2\).

\[ f(x) = 6 - x^2 \quad \text{and} \quad g(x) = |x| \]

By symmetry

\[
\int_{-2}^{2} \left[ 6-x^2 - |x| \right] \, dx = 2 \int_{0}^{2} \left[ (6-x^2-x) \, dx \right] = 2 \int_{0}^{6} \left[ -x^2-x+6 \right] \, dx
\]

\[
= 2 \left[ \frac{-1}{3} x^3 - \frac{1}{2} x^2 + 6x \right]_{0}^{2} = 2 \left[ \frac{-1}{3} (8) - \frac{1}{2} (4) + 6(2) - 0 \right]
\]

\[
= 2 \left[ -\frac{8}{3} - 2 + 12 \right] = 2 \left[ \frac{-8 + 30}{3} \right] = 2 \left[ \frac{22}{3} \right]
\]

\[ = \frac{44}{3} \]

The area of the region is \( \frac{44}{3} \).
6. Use the Comparison Test to show whether the following converges or diverges.

\[ \int_{1}^{\infty} \frac{1}{x + e^{2x}} \, dx \]

Check:

\[ \int_{1}^{\infty} \frac{1}{e^{2x}} \, dx = \int_{1}^{\infty} e^{-2x} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{2} e^{-2x} \right]_{1}^{t} = \frac{1}{2} e^{-2} \]

Since \( e^{2x} \leq x + e^{2x} \) for \( x \geq 1 \)

\[ \frac{1}{x + e^{2x}} \leq \frac{1}{e^{2x}} \] for \( x \geq 1 \)

By the Comparison Test, since \( \int_{1}^{\infty} \frac{1}{e^{2x}} \, dx \) converges,

\[ \int_{1}^{\infty} \frac{1}{x + e^{2x}} \, dx \] converges.
7. Compute the volume of the solid created when the region bounded by \( y = x \sqrt{x - 1}, \)
\( y = 0, \) and \( z = 5 \) is rotated around the \( x-\)axis.

Show Your Work.

\[
\text{Method of Disks}
\]

\[
\pi \left[ f(x) \right]^2 = \pi \left[ x \sqrt{x-1} \right]^2
\]

\[
= \pi \left( x \right) \left( x-1 \right) = \pi \left( x^2 - x^2 \right)
\]

\[
V = \int_1^5 \pi (x^2 - x^2) \, dx = \pi \left[ \frac{1}{4} x^4 - \frac{1}{3} x^3 \right]_1^5
\]

\[
= \pi \left[ \left( \frac{1}{4} (625) - \frac{1}{3} (125) \right) - \left( \frac{1}{4} (1) - \frac{1}{3} (1) \right) \right]
\]

\[
= \pi \left[ \frac{625}{4} - \frac{125}{3} - \frac{1}{4} + \frac{1}{3} \right] = \left( \frac{624}{4} - \frac{124}{3} \right) \pi
\]

\[
= \frac{344}{3} \pi
\]

The volume is \( \frac{344}{3} \pi. \)
8. For the following function over the interval $1 \leq x \leq 6$, consider the Midpoint Rule with $n = 50$.

$$f(x) = \frac{1}{\sqrt{x} + 2}$$

Fill out the information below.

$$\Delta x = \frac{6-1}{50} = \frac{5}{50} = \frac{1}{10} = 0.1$$

$$x_i = 1 + (i - \frac{1}{2})(0.1)$$

Write out the summation for the Midpoint sum. You must use summation notation.

$$M_{50} = \sum_{i=1}^{50} \frac{1}{\sqrt{1 + (i - 0.5)(0.1)} + 2} \times (0.1)$$

Use your calculator to find the value of the Midpoint sum. Give answer to 6 decimal places.

The calculator gives,

$$M_{50} \approx 2.192722$$
9. Suppose \( f(x) = e^{x^2} \) over \( 0 \leq x \leq 2 \), with \( n = 40 \).

For the Left sum find the following.

\[
\Delta x = \frac{2-0}{40} = \frac{1}{20} \approx 0.05
\]

\[
x_i^* = 0 + (i-1)(0.05)
\]

Use your calculator to compute the Left sum.
(Give answer to 6 decimal places.)

\[
L_{40} = \sum_{i=1}^{40} e^{((i-1)0.05)^2} (0.05) \approx 15.158131
\]

Use your calculator to compute the Right sum.
(Give answer to 6 decimal places.)

\[
R_{40} = \sum_{i=1}^{40} e^{(0.05i)^2} (0.05) \approx 17.838038
\]

Use \( T_{40} = \frac{L_{40} + R_{40}}{2} \) to get the Trapezoidal sum. Give 6 decimal places.

\[
T_{40} = \frac{L_{40} + R_{40}}{2} \approx 16.498085
\]