Different students master different skills at different speeds. No two students learn exactly the same way at the same time. However, at some point you will be expected to perform certain skills with good accuracy. In Algebra 1, your teacher did not want you to grab for your calculator if you needed to calculate \(-16 + 9\) or \(\sqrt{81}\). The same thing is true of your Algebra 2 teacher. There are certain algebra skills that you need to be able to do on your own and with good accuracy. Most of the Milepost problems are skills that you should have been developing in Algebra 1 and Geometry. If you have not mastered these skills by now it does not necessarily mean that you will not be successful in this class. However, it does mean that to be successful you are going to need to do some extra work outside of class. You need to get caught up on the algebra skills that this year's teacher and possibly next year's pre-calculus teacher expect.

Starting in Unit 2 and finishing in Unit 9, there are twenty one problems designated as Milepost problems. Each one has an icon like the one above. After you do each of the Milepost problems, check your answers. If you have not done them correctly, this is your reminder that you need to put in some extra practice on that skill. The following practice sets are keyed to each of the Milepost problems in the textbook. Each of the Milepost practice sets has the topic clearly labeled, followed by some completed examples. Next, the solution to the Milepost problem from the book is given. Following that are more problems to practice with answers included.

Warning! Looking is not the same as doing. You will never become good at any sport just by watching it. In the same way, reading through the worked out examples and understanding the steps are not the same as being able to do the problems yourself. An athlete only gets good with practice. The same thing is true of your algebra skills. If you did not get the Milepost problem correct you need the extra practice. How many of the extra practice problems do you need to try? That is really up to you. Remember that your goal is to be able to do that skill on your own confidently and accurately. Another warning! You should not expect your teacher to spend time in class going over the solutions to the Milepost practice sets. After reading the examples and trying the problems, if you still are not successful, talk to your teacher about getting a tutor or extra help outside of class time.

Two other sources for help with the Milepost problems and the other new topics in Algebra 2 are the Parent's Guide with Review to Math 1 (Algebra 1) and the Parent's Guide with Review to Math 3 (Algebra 2). Information about ordering these supplements can be found inside the front page of the student text. These resources are also available free from the internet at www.cpm.org.

**Milepost Topics**

1. x- and y-intercepts of a quadratic equation
2. Graphing lines using slope and y-intercept
3. Simplifying expressions with exponents
4. Solving systems of linear equations
5. Multiplying polynomials
6. Factoring quadratic expressions
7. Solving multi-variable equations
8. Slope of the line and distance
9. Function notation; domain and range
10. Writing equations of lines given two points
11. Slopes of parallel and perpendicular lines
12. Graphing linear inequalities
13. Multiplication and division of rational expressions
14. Solving rational equations
15. Addition and subtraction of rational expressions
16. Integral and rational exponents
17. Completing the square; locator points for parabolas
18. Absolute value equations and inequalities
19. Writing and solving exponential equations
20. Finding the equation for the inverse of a function
21. Solving a system of equations in three variables
The y-intercept of a equation is the location where the graph crosses the y-axis. To find the y-intercept of an equation, substitute x = 0 into the equation and solve for y. For example:

Find the y-intercept for the equation \( y = x^2 + 4x - 12 \).
If \( x = 0 \), then \( y = (0)^2 + 4(0) - 12 = -12 \). The y-intercept is (0, -12).

The x-intercept of a equation is the location where the graph crosses the x-axis. To find the x-intercept of an equation, substitute y = 0 into the equation and solve for x by factoring or using the quadratic formula. Here are two examples:

Find the x-intercept for the equation \( y = x^2 + 4x - 12 \).
If \( y = 0 \), then

\[
0 = x^2 + 4x - 12
\]

By factoring and using the zero product property
\[
x = -6 \text{ or } x = 2
\]
The x-intercepts are (-6, 0) and (2, 0)

Find the x-intercept for the equation \( y = 2x^2 - 3x - 3 \).
If \( y = 0 \), then

\[
0 = 2x^2 - 3x - 3
\]

Since we cannot factor the trinomial we use the Quadratic Formula to solve for x. If \( ax^2 + bx + c = 0 \), then

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

substitute for a, b, and c
\[
x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)}
\]
simplify and add
\[
= \frac{3 \pm \sqrt{9 + 24}}{4} = \frac{3 \pm \sqrt{33}}{4}
\]

find \( \sqrt{33} \) value
\[
\approx \frac{3 \pm 5.745}{4} \text{ or } \frac{3 \pm 5.745}{4} \text{ and } \frac{3 \pm 5.745}{4}
\]
simplify the fractions
and the x-intercepts are approximately (2.19, 0) and (-0.69, 0).
Now we can go back and try the original question. Find the x- and y-intercepts for the graph of:

\[ y = x^2 + 4x - 17. \]

To find the y-intercept let \( x = 0 \) so \( y = (0)^2 + 4(0) - 17 = -17 \).

To find the x-intercept let \( y = 0 \) so \( 0 = x^2 + 4x - 17. \)

Since we cannot factor we use the Quadratic Equation with \( a = 1, b = 4, \) and \( c = -17 \)

\[
\begin{align*}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{16 + 68}}{2} \\
&= \frac{-4 \pm \sqrt{84}}{2} \\
&= \frac{-4 \pm 2\sqrt{21}}{2} \\
&= -2 \pm \sqrt{21},
\end{align*}
\]

The answers are \((0, -17), (\pm \sqrt{21}, 0)\) or \((2.58, 0), (-6.58, 0)\)

Here are some more to try. Find the x- and y-intercepts for each equation,

1) \( y = 2x^2 - 9x - 35 = 0 \)
2) \( y = 2x^2 - 11x + 5 \)
3) \( 3x^2 + 2 + 7x = y \)
4) \( 8x^2 + 10x + 3 = y \)
5) \( y + 2 = x^2 - 5x \)
6) \( (x - 3)(x + 4) - 7x = y \)
7) \( -4x^2 + 8x + 3 = y \)
8) \( 0.09x^2 - 0.86x + 2 = y \)
9) \( y = 2x^3 - 50x \)
10) \( y = 3x^2 + 4x \)

**Answers:**

Note that the coordinates of the intercepts are \((x, 0)\) and \((0, y)\).

1. \( x = 7, \frac{-5}{2} \) \( y = -35 \)  
2. \( x = 5, \frac{1}{2} \) \( y = 5 \)
3. \( x = -\frac{1}{3}, -2 \) \( y = 2 \)  
4. \( x = -\frac{3}{4}, -\frac{1}{2} \) \( y = 3 \)
5. \( x = (5 \pm \sqrt{33})/2, \approx 5.37, -0.37 \) \( y = -2 \)  
6. \( x = (6 \pm \sqrt{84})/2 \approx 7.58, -1.58 \) \( y = -12 \)
7. \( x = (-8 \pm \sqrt{128})/8 \approx -0.32, 2.32 \) \( y = 3 \)  
8. \( x = 5.56, 4 \) \( y = 2 \)
9. \( x = 0, 5, -5 \) \( y = 0 \)  
10. \( x = 0, -\frac{4}{3} \) \( y = 0 \)
Graphing a Line using Slope and Y-Intercept

If an equation of a line is written in the form $y = mx + b$, then the y-intercept is the point $(0, b)$. The slope of the line is the coefficient of $x$, represented in the general form of the equation as $m$. So in the equation $y = \frac{2}{3}x + 7$, the slope is $\frac{2}{3}$ and the y-intercept is $(0, 7)$.

Let's first see how to use the information in slope-intercept equations to graph a line.

Without making a table. graph each line. Start with the y-intercept, then use the slope.

a) $y = \frac{2}{3}x - 2$

In part (a), we start by identifying the slope and y-intercept. The slope is $\frac{2}{3}$ and the y-intercept is $(0, -2)$. To graph the line we plot the y-intercept. (Before continuing, imagine what the line will look like.) The fact that the slope is positive tells us the direction of the line is upward left to right. Then, knowing that the slope is $\frac{2}{3}$, we can find another point on the line by starting at the y-intercept, moving our pencil up vertically two units and then horizontally (to the right) three units. Just remember that the slope is positive! After moving vertically 2 units and horizontally 3 units, we arrive at another point on the line.

b) $y = 4 - 2x$

In part (b), $y = 4 - 2x$, don't let the form of the equation fool you. The slope is $-2$ or $-\frac{2}{1}$ and the y-intercept is $(0, 4)$. The slope is always the coefficient of $x$ and the y-intercept is always the constant. Rearranging their order doesn't change their meaning.

If the equation is not already in slope/intercept (y-form) then the equation must first be solved for $y$. If the equation to graph is $2x + 5y = 10$, then after solving we get $y = \frac{10 - 2x}{5}$ or $y = -\frac{2}{5}x + 2$ so the y-intercept is $(0, 2)$ and the slope is $-\frac{2}{5}$. 
Two other special cases to remember are vertical and horizontal lines.

\[ y = 2 \] is a horizontal line (slope equal to zero) passing through the \( y \)-axis at \((0, 2)\).

\[ x = 3 \] is a vertical line (undefined slope) passing through the \( x \)-axis at \((3, 0)\).

Now we can go back to the original question.  
Graph each line and find the intersection.

\[
\begin{align*}
  x + y &= 5 \\
  y &= \frac{1}{3} x + 1 
\end{align*}
\]

The first equation needs to be solved for \( y \): \( y = -x + 5 \) so the \( y \)-intercept is \((0, 5)\) and the slope is -1 or \(-\frac{1}{1}\).  The second equation has a \( y \)-intercept of \((0, 1)\) and a slope of \( \frac{1}{3} \).  After graphing the two lines you see that they intersect at the point \((3, 2)\).

Here are some more to try.  Use the slope and \( y \)-intercept to graph each line and tell the point of intersection.

1) \[
\begin{align*}
  y &= -x + 8 \\
  y &= x - 2 
\end{align*}
\]

2) \[
\begin{align*}
  y &= -x + 3 \\
  y &= x + 3 
\end{align*}
\]

3) \[
\begin{align*}
  y &= -3x \\
  y &= -4x + 2 
\end{align*}
\]

4) \[
\begin{align*}
  y &= -x + 5 \\
  y &= \frac{1}{2} x + 2 
\end{align*}
\]

5) \[
\begin{align*}
  y &= -2x - 1 \\
  y &= -4x + 3 
\end{align*}
\]

6) \[
\begin{align*}
  3x + 3y &= 4 + x \\
  4 - 2x &= 3y 
\end{align*}
\]

7) \[
\begin{align*}
  y &= 2 \\
  2x + y &= 4 
\end{align*}
\]

8) \[
\begin{align*}
  x &= 3 \\
  2x + 3y &= 0 
\end{align*}
\]

9) \[
\begin{align*}
  2x + 3y &= 0 \\
  2x - 3y &= 0 
\end{align*}
\]

10) \[
\begin{align*}
  3x - 2y &= 4 \\
  2y &= 3x - 6 
\end{align*}
\]

\textbf{Answers:}

1. \((5, 3)\)  
2. \((0, 3)\)  
3. \((2, -6)\)  
4. \((2, 3)\)  
5. \((2, -5)\)  
6. same line  
7. \((1, 2)\)  
8. \((3, -2)\)  
9. \((0, 0)\)  
10. no solution-parallel lines
There are three basic patterns for expressions with positive exponents. They are summarized below with some examples.

1) \(x^a \cdot x^b = x^{a+b}\)
   examples: \(x^3 \cdot x^4 = x^{3+4} = x^7;\) \(2^7 \cdot 2^4 = 2^{11}\)

2) \(\frac{x^a}{x^b} = x^{a-b}\)
   examples: \(x^{10} \div x^4 = x^{10-4} = x^6;\) \(\frac{2^4}{2^7} = 2^{-3}\)

3) \((x^a)^b = x^{ab}\)
   examples: \((x^4)^3 = x^{4 \cdot 3} = x^{12};\) \((2x^3)^4 = 2^4 \cdot x^{12} = 16x^{12}\)

Now we can go back and try the original problem. Simplify each expression.

a) \((2x^2y)^4 = 2^4x^8y^4 = 16x^8y^4\)

b) \(\frac{-3x^2y^3}{(6x)^2} = \frac{-3x^2y^3}{36x^2} = \frac{y^3}{12}\)

c) \(\frac{(2x^2y^4)}{3xy^5} = \frac{16x^8y^4}{3xy^5} = \frac{16x^7}{3y}\)

d) \(5(5xy)^2(x^3y) = 5(25x^2y^2)(x^3y) = 125x^5y^3\)

Here are some more to try. Use the properties of exponents to write each of the following expressions in a simpler form.

1. \(3x^2 \cdot x\)
2. \(\frac{n^{12}}{n^3}\)
3. \((x^3)^2\)
4. \((-2x^2)(-2x)\)
5. \(\frac{-8x^6y^2}{-4xy}\)
6. \((2x^3)^3\)
7. \((10^3)^4\)
8. \(3^2 \cdot 3^5\)
9. \(10^5 \div 10^3\)
10. \(x^2y^3 \cdot x^3y^4\)
11. \((x^3)^4\)
12. \(\frac{6x^2y^3}{2xy}\)
13. \(-3x^2 \cdot 4x^3\)
14. \((2x^2)^3\)
15. \((x^2y)^2(2x)^3\)
16. \(\frac{m^{16}y^{31}}{m^2y^{17}}\)
17. \((6x^3z)^3\)
18. \(\left(3x^2\right)^2 + (6x^4)\)
19. \((5x)^2(3y)^3\)
20. \(\left(3x^{11}z^5\right)^2\)
21. \((2b)^5(3k)^2\)
22. \(\frac{3x^2z^3}{6x^3}\)
23. \(\left(6x\right)^2 + \left(24x^3\right)\)
24. \(\frac{6x^2y^3}{2xy}\)
Milepost Number 4
FX-36
Solving Systems of Linear Equations in Two Variables

You can solve systems of equations with a variety of methods. You can graph, use the Substitution Method, or the Elimination Method. Each method works best with certain forms of equations. Here are some examples and then we can return to the original problem.

For each system below, determine which method would be best (easiest) to use. Then solve the system to find the point of intersection.

a) \[ x = 4y - 7 \]
\[ 3x - 2y = 1 \]

b) \[ y = \frac{3}{4}x - 1 \]
\[ y = -\frac{1}{3}x - 1 \]

c) \[ x + 2y = 16 \]
\[ x - y = 2 \]

d) \[ x + 3y = 4 \]
\[ 3x - y = 2 \]

Although the method that is easiest for one person may not be the easiest for another, the most common methods are shown on the next two pages. You should use the method you are comfortable with and with which you are most successful.
a) \[ x = 4y - 7 \]
\[ 3x - 2y = 1 \]

**Substitution:** Substitute \( 4y - 7 \) for \( x \) in the second equation.

\[
\begin{align*}
3(4y - 7) - 2y &= 1 \\
12y - 21 - 2y &= 1 \\
10y &= 22 \\
y &= \frac{22}{10} = 2.2
\end{align*}
\]

Find \( x \):
\[
\begin{align*}
x &= 4(2.2) - 7 \\
x &= 8.8 - 7 = 1.8
\end{align*}
\]

Solution: (1.8, 2.2)

b) \[ y = \frac{3}{4} x - 1 \]
\[ y = -\frac{1}{3} x - 1 \]

**Graphing:** Normally graphing is not the best way to solve a system of equations, but since both equations are in \( y \)-form and if you happened to notice that they have the same \( y \)-intercept, you can tell that they cross at (0, -1) the \( y \)-intercept. We did not actually graph here, but we used the principles of the graphs to solve the system of equations. Substitution will work nicely as well.

Solution: (0, -1)

c) \[ x + 2y = 16 \]
\[ x - y = 2 \]

**Elimination:** Subtract the second equation from the first.

\[
\begin{align*}
0 + 3y &= 14 \\
3y &= 14 \\
y &= \frac{14}{3}
\end{align*}
\]

Find \( x \) by substituting \( y = \frac{14}{3} \) into the second equation:
\[
\begin{align*}
x - \frac{14}{3} &= 2 \\
x &= 2 + \frac{14}{3} = \frac{20}{3}
\end{align*}
\]

Solution: \( \left(\frac{20}{3}, \frac{14}{3}\right) \)

d) \[ x + 3y = 4 \]
\[ 3x - y = 2 \]

**Elimination with a multiplication first.** Multiply the bottom equation by 3 and add it to the top equation.

\[
\begin{align*}
x + 3y &= 4 \\
+ 9x - 3y &= 6 \\
10x &= 10 \\
x &= 1
\end{align*}
\]

Find \( y \) by substituting \( x = 1 \) into the second equation:
\[
\begin{align*}
3(1) - y &= 2 \\
3 - y &= 2 \\
y &= 1
\end{align*}
\]

Solution: (1, 1)

Now we can go back and look at the original problem.

Solve this system of linear equations in two variables:
\[
\begin{align*}
5x - 4y &= 7 \\
2y + 6x &= 22
\end{align*}
\]

You may use substitution or elimination but both methods need a little work to get started.
**Substitution method:**

Before we can substitute we need to isolate one of the variables. Solve the second equation for \( y \) and it becomes

\[
y = 11 - 3x.
\]

Now substitute \( 11 - 3x \) for \( y \) in the first equation and solve.

\[
5x - 4(11 - 3x) = 7
\]
\[
5x - 44 + 12x = 7
\]
\[
17x - 44 = 7
\]
\[
x = 3
\]

Solve for \( y \):
\[
y = 11 - 3(3) = 2
\]
Solution (3, 2)

**Elimination method:**

Before we can eliminate we need to rearrange the second equation so that the variables line up.

\[
5x - 4y = 7
\]
\[
6x + 2y = 22
\]

Now multiply the second equation by 2 and add to eliminate \( y \).

\[
10x - 8y = 14
\]
\[
+ 12x + 4y = 44
\]
\[
17x = 51
\]
\[
x = 3
\]

Solve for \( y \) in the first equation:
\[
5(3) - 4y = 7
\]
\[
-4y = -8
\]
\[
y = 2
\]
Solution (3, 2)

Here are some more to try. Find the solution to these systems of linear equations. Use the method of your choice.

\[
1. \quad \begin{align*}
y &= 3x - 1 \\
2x - 3y &= 10
\end{align*}
\]
\[
2. \quad \begin{align*}
x &= \frac{1}{2} y + 4 \\
8x + 3y &= 31
\end{align*}
\]
\[
3. \quad \begin{align*}
2y &= 4x + 10 \\
6x + 2y &= 10
\end{align*}
\]
\[
4. \quad \begin{align*}
3x - 5y &= -14 \\
x + 5y &= 22
\end{align*}
\]
\[
5. \quad \begin{align*}
4x + 5y &= 11 \\
2x + 6y &= 16
\end{align*}
\]
\[
6. \quad \begin{align*}
x + 2y &= 5 \\
x + y &= 5
\end{align*}
\]
\[
7. \quad \begin{align*}
y &= 2x - 3 \\
x - y &= -4
\end{align*}
\]
\[
8. \quad \begin{align*}
y + 2 &= x \\
3x - 3y &= x + 14
\end{align*}
\]
\[
9. \quad \begin{align*}
2x + y &= 7 \\
x + 5y &= 12
\end{align*}
\]
\[
10. \quad \begin{align*}
y &= \frac{3}{5} x - 2 \\
y &= \frac{x}{10} + 1
\end{align*}
\]
\[
11. \quad \begin{align*}
2x + y &= -2x + 5 \\
3x + 2y &= 2x + 3y
\end{align*}
\]
\[
12. \quad \begin{align*}
4x - 3y &= -10 \\
x &= \frac{1}{4} y - 1
\end{align*}
\]

**Answers:**

1. (-1, -4) 2. (7/2, 1) 3. (0, 5) 4. (2, 4) 5. (-1,3) 6. (5, 0)

7. (7, 11) 8. (-8, -10) 9. (23/9, 17/9) 10. (6, 1.6) 11. (1, 1) 12. (7/4, 3)
The product of polynomials can be found by using the distributive property or using generic rectangles. If you are multiplying two binomials, you can also use the F.O.I.L. method.

Let us look at an example for each of the three methods before returning to the original problem.

In multiplying binomials, such as \((3x - 2)(4x + 5)\), you might use a generic rectangle.

\[
\begin{array}{c|c}
4x & 3x & -2(4x) \\
5 & 3x(5) & -2(5) \\
\end{array}
\Rightarrow \begin{array}{c|c}
12x^2 & -8x \\
15 & -10 \\
\end{array}
\]

= \(12x^2 + 7x - 10\)

You might view multiplying binomials with generic rectangles as a form of double distribution. The \(4x\) is distributed across the first row of the generic rectangle and then the \(5\) is distributed across the second row of the generic rectangle. Some people write it this way:

\[
(3x - 2)(4x + 5) = (3x - 2)(4x) + (3x - 2)(5) = 12x^2 - 8x + 15x - 10
\]

= \(12x^2 + 7x - 10\).

Another approach to multiplying binomials is to use the mnemonic ‘F.O.I.L.’ F.O.I.L. is an acronym for First, Outside, Inside, Last:

- **F.** multiply the FIRST terms of each binomial
  \((3x)(4x) = 12x^2\)
- **O.** multiply the OUTSIDE terms
  \((3x)(5) = 15x\)
- **I.** multiply the INSIDE terms
  \((-2)(4x) = -8x\)
- **L.** multiply the LAST terms of each binomial
  \((-2)(5) = -10\)

Finally, we combine like terms: \(12x^2 + 15x - 8x - 10 = 12x^2 + 7x - 10\).
Now we can go back and try the original problem again using a variety of methods.

Multiply and simplify.

a) \((x + 1)(2x^2 - 3)\)

We can use F.O.I.L. here.

\[
\begin{array}{c|c|c|c|c}
F & O & I & L \\
\hline
(x)(2x^2) & (x)(-3) & (1)(2x^2) & (1)(-3) \\
\hline
2x^3 & -3x & 2x^2 & -3 \\
\hline
\end{array}
\]

\[2x^3 - 3x + 2x^2 - 3 = 2x^3 + 2x^2 - 3x -3\]

b) \((x + 1)(x - 2x^2 + 3)\)

Using generic rectangles

\[
\begin{array}{c|c|c|c}
& x^2 & -2x & 3 \\
\hline
x^2 & & & \\
+ & -2x & 3 \\
\hline
1 & & & \\
\hline
\end{array}
\]

\[x^3 - x^2 + x + 3\]

c) \(2(x + 3)^2\)

Write our the factors and distribute.

\[2(x + 3)(x + 3) = (2x + 6)(x + 3)\]

\[(2x + 6)(x) + (2x + 6)(3)\]

\[2x^2 + 6x + 6x + 18 = 2x^2 + 12x + 18\]

d) \((x + 1)(2x - 3)^2\)

Write out the factors. Multiply two of the factors together and then multiply that answer by the third factor.

\[(x + 1)(2x - 3)(2x - 3)\]

\[(2x^2 - x - 3)(2x - 3)\]

\[4x^3 - 6x^2 - 2x^2 + 3x - 6x + 9\]

\[4x^3 - 8x^2 - 3x + 9\]

Here are some more to try. Multiply and simplify.

1. \((2x + 3)(x - 7)\)
2. \((4x - 2)(3x + 5)\)
3. \((x - 2)(x^2 + 3x + 5)\)
4. \((x + 8)(x - 12)\)
5. \(4(3x - 5)^2\)
6. \((2x + y)(2x - y)\)
7. \((2x + 3)^2\)
8. \((5x - 8)(2x + 7)\)
9. \((x + 3)(x^2 - 4x + 7)\)
10. \((x + 7)(x - 11)\)
11. \(-8x^3(5x^2 + 7)\)
12. \((2x + y)(x + 1)^2\)

Answers:

1. \(2x^2 - 11x - 21\)
2. \(12x^2 + 14x - 10\)
3. \(x^3 + x^2 - x - 10\)
4. \(x^2 - 4x - 96\)
5. \(36x^2 - 120x + 100\)
6. \(4x^2 - y^2\)
7. \(4x^2 + 12x + 9\)
8. \(10x^2 + 19x - 56\)
9. \(x^3 - x^2 - 5x + 21\)
10. \(x^2 - 4x - 77\)
11. \(-40x^5 - 56x^3\)
12. \(2x^3 + 4x^2 + 2x + x^2y + 2xy + y\)
Factoring Quadratic Expressions

Factoring quadratics means changing the expression into a product of factors or to find the dimensions of the generic rectangle. You can use diamond problems with generic rectangles or just guess and check with F.O.I.L. or the distributive property. Here are some examples using diamonds and generic rectangles:

Diamond Problems can be used to help factor easier quadratics like \( x^2 + 6x + 8 \).

\[
\begin{array}{c|c}
2 & 8 \\
6 & 4 \\
\end{array}
\begin{array}{c|c}
\text{x}^2 & 4x \\
2x & 8 \\
\end{array}
\begin{array}{c|c}
x + 4 & x \\
2 & x + 4 \\
\end{array}
\Rightarrow (x + 4)(x + 2)
\]

We can modify the diamond method slightly to factor problems that are a little different in that they no longer have a “1” as the coefficient of \( x^2 \). For example, factor:

\[
2x^2 + 7x + 3
\]

\[
\begin{array}{c|c}
? & 7 \\
? & 6 \\
\end{array}
\begin{array}{c|c}
2x^2 & 6x \\
1x & 3 \\
\end{array}
\begin{array}{c|c}
x + 3 & x + 3 \\
1 & 1 \\
\end{array}
\Rightarrow (2x + 1)(x + 3)
\]

Another problem: \( 5x^2 - 13x + 6 \). Note that the upper value in the diamond is the product of 5 and 6.

\[
\begin{array}{c|c}
-3 & -10 \\
-13 & -30 \\
\end{array}
\begin{array}{c|c}
5x^2 & -10x \\
-3x & 6 \\
\end{array}
\begin{array}{c|c}
5x & -2 \\
5x^2 & -10x \\
3 & -6 \\
\end{array}
\Rightarrow (5x - 3)(x - 2)
\]

Now we can go back and try the original problem. Factor each quadratic.

\[
\begin{array}{c|c}
0 & -4 \\
? & ? \\
\end{array}
\begin{array}{c|c}
2 & -4 \\
0 & 2 \\
\end{array}
\begin{array}{c|c}
4x^2 & -2x \\
2x & -1 \\
\end{array}
\begin{array}{c|c}
2x & -1 \\
4x^2 & -2x \\
1 & -1 \\
\end{array}
\Rightarrow (2x + 1)(2x - 1)
\]

\[
\begin{array}{c|c}
-4 & ? \\
? & 0 \\
\end{array}
\begin{array}{c|c}
2 & 0 \\
-4 & -2 \\
\end{array}
\begin{array}{c|c}
4x^2 & -2x \\
2x & -1 \\
\end{array}
\begin{array}{c|c}
2x & -1 \\
4x^2 & -2x \\
1 & -1 \\
\end{array}
\Rightarrow (2x + 1)(2x - 1)
\]
Here are some more to try. Factor each expression

1. \(2x^2 + 7x - 4\)  
2. \(7x^2 + 13x - 2\)  
3. \(3x^2 + 11x + 10\)  
4. \(x^2 + 5x - 24\)  
5. \(2x^2 + 5x - 7\)  
6. \(3x^2 - 13x + 4\)  
7. \(64x^2 + 16x + 1\)  
8. \(5x^2 + 12x - 9\)  
9. \(8x^2 + 24x + 10\)  
10. \(6x^3 + 31x^2 + 5x\)

**Answers:**

1. \((x + 4)(2x - 1)\)  
2. \((7x - 1)(x + 2)\)  
3. \((3x + 5)(x + 2)\)  
4. \((x + 8)(x - 3)\)  
5. \((2x + 7)(x - 1)\)  
6. \((3x - 1)(x - 4)\)  
7. \((8x + 1)^2\)  
8. \((5x - 3)(x + 3)\)  
9. \(2(4x^2 + 12x + 5) = 2(2x + 1)(2x + 5)\)  
10. \(x(6x^2 + 31x + 5) = x(6x + 1)(x + 5)\)
Solving for One Variable in an Equation
or Formula with Two or More Variables

When you solve for one variable in an equation with two or more variables it usually helps to start by simplifying, for example, removing parentheses and fractions. Then isolate the desired variable in the same way as you solve an equation with only one variable. Here are two examples.

Solve for y: \( \frac{x - 3(y + 2)}{4} + 2(x + 1) = 7 \)

First multiply all terms by 4 to remove the fractions and then simplify.

\[
\begin{align*}
(4) \frac{x - 3(y + 2)}{4} + (4)2(x + 1) &= (4)7 \\
x - 3y + 8x + 8 &= 28 \\
9x - 3y &= 20
\end{align*}
\]

Then solve for y.

\[
y = \frac{-9x + 20}{3} = 3x - \frac{20}{3}
\]

Solve for y: \( x + 2\sqrt{y + 1} \) = 3x + 4

First isolate the radical.

\[
\begin{align*}
x + 2\sqrt{y + 1} &= 3x + 4 \\
2\sqrt{y + 1} &= 2x + 4 \\
\sqrt{y + 1} &= x + 2
\end{align*}
\]

Then remove the radical by squaring both sides. Remember \((x + 2)^2 = (x+2)(x+2)\).

\[
(\sqrt{y + 1})^2 = (x + 2)^2 \\
y + 1 = x^2 + 4x + 4 \\
y = x^2 + 4x + 3
\]

Now we can go back and look at the original problem.

Rewrite the following equations so that you could enter them into the graphing calculator. In other words, solve for y.

a) \( x - 3(y + 2) = 6 \)
\( x - 3y - 6 = 6 \)
\( x - 3y = 12 \)
\( -3y = -x + 12 \)
\( y = \frac{-x + 12}{3} \) or \( y = \frac{1}{3}x - 4 \)

b) \( \frac{6x^2 - 11}{y} = 2 \)
\( \frac{6x^2 - 11}{y} = 5 \)
\( \frac{y}{6x^2 - 11} = \frac{1}{5} \)
\( 6x - \frac{11}{5} = 5y \)
\( y = \frac{6x^2 - 11}{5} \) or \( y = \frac{6}{5}x - \frac{11}{5} \)

c) \( \sqrt{y + 14} = x + 1 \)
\( (\sqrt{y + 14})^2 = (x + 1)^2 \)
\( y - 4 = (x + 1)^2 \)
\( y = (x + 1)^2 + 4 \) or \( x^2 + 2x + 5 \)

d) \( \sqrt{y + 14} = x + 2 \)
\( (\sqrt{y + 14})^2 = (x + 2)^2 \)
\( y + 4 = x^2 + 4x + 4 \)
\( y = x^2 + 4x \)
Here are some more to try. Solve for y.

1. \(2x - 5y = 7\)  
2. \(2(x + y) + 1 = x - 4\)
3. \(4(x - y) + 12 = 2x - 4\)  
4. \(x = \frac{1}{3}y - 2\)
5. \(x = y^2 + 1\)  
6. \(\frac{5x!+!2}{y} - 1 = 5\)
7. \(\sqrt{y!+!3} = x - 2\)  
8. \((y + 2)^2 = x^2 + 9\)
9. \(\frac{x!+!2}{4} + \frac{4!+!y}{y} = 3\)  
10. \(\sqrt{2y!+!1} = x + 3\)
11. \(\frac{2}{4!+!y}\)  
12. \(\frac{y!+!1}{y!-!1}\)

Answers:

1. \(y = \frac{2}{5}x - \frac{7}{5}\)  
2. \(y = \frac{1}{2}x + \frac{5}{2}\)  
3. \(y = \frac{1}{2}x + 4\)  
4. \(y = 5x + 10\)  
5. \(y = \pm \sqrt{x^2!+!1}\)
6. \(y = \frac{5}{6}x + \frac{1}{3}\)  
7. \(y = x^2 - 4x + 1\)  
8. \(y = \pm \sqrt{x^2!+!9} - 2\)  
9. \(y = \frac{1}{2}x - 1\)
10. \(y = \frac{1}{2}x^2 + 3x + 4\)  
11. \(y = \frac{4x!+!12}{x}\)  
12. \(y = \frac{x!+!1}{y!-!1}\)
Find the Slope of the Line Through Two Given Points and the Distance Between the Two Points

To compute either the slope or the distance determined by two points, a generic right triangle provides a good diagram. The slope is the ratio of the vertical leg over the horizontal leg. (Remember to check whether it is negative or positive.) The distance is the length of the hypotenuse which is found by using the Pythagorean theorem. Here are two examples.

Use a generic triangle to find the slope and the distance between the given points.

a) \((-2, 3)\) and \((3, 5)\)

\[
\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{2}{5}.
\]

The distance is found by \(d^2 = 2^2 + 5^2 = 29\).

So the distance \(d = \sqrt{29} \approx 5.39\).

A point moving from left to right along this line moves up, a positive direction.

b) \((-7, 20)\) and \((3, -5)\)

\[
\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = -\frac{25}{10} = -\frac{5}{2}.
\]

The distance is found by \(d^2 = 25^2 + 10^2 = 725\).

So the distance \(d = \sqrt{725} \approx 26.92\).

A point moving along this line from left to right moves down in a negative direction.
We can now go back and try the original problem. Use a generic triangle to find the slope of the line through the two given points and then find the distance between the two points.

\[ \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} \]

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\( a) \quad \text{b) } \]

\[ \text{slope} = \frac{4}{4} = 1 \]
\[ d = \sqrt{4^2 + 4^2} = \sqrt{32} \approx 5.66 \]

\[ \text{slope} = \frac{3}{6} = \frac{1}{2} \]
\[ d = \sqrt{6^2 + 13^2} = \sqrt{155} \approx 6.71 \]

\( c) \quad \text{d) } \]

Sometimes people like to use formulas that represent the diagram. Using the points \((12, 18)\) and \((-16, -19)\):

\[ \text{slope} = \frac{18 - (-19)}{12 - (-16)} = \frac{17}{6} \]
\[ \text{distance} = \sqrt{(-28)^2 + (28)^2} = \sqrt{980} \approx 31.31 \]

\( y_2 - y_1 \) and \( x_2 - x_1 \) represent the lengths of the vertical and horizontal legs respectively.

Here are some more to try. Find the slope of the line through the two given points and then find the distance between the two points.

1. \((1, 2)\) and \((4, 5)\)  
2. \((7, 3)\) and \((5, 4)\)  
3. \((-6, 8)\) and \((-4, 5)\)
4. \((5, 0)\) and \((0, 1)\)  
5. \((10, 2)\) and \((2, 24)\)  
6. \((-6, 5)\) and \((8, -3)\)
7. \((-3, 5)\) and \((2, 12)\)  
8. \((-6, -3)\) and \((2, 10)\)  
9. \((-15, 39)\) and \((29, -2)\)

**Answers:**

1. \( m = 1 \quad d = \sqrt{18} \)  
2. \( m = \frac{1}{2} \quad d = \sqrt{5} \)  
3. \( m = \frac{3}{2} \quad d = \sqrt{13} \)

4. \( m = \frac{1}{5} \quad d = \sqrt{26} \)  
5. \( m = \frac{11}{4} \quad d = \sqrt{548} \)  
6. \( m = \frac{4}{3} \quad d = \sqrt{260} \)

7. \( m = \frac{7}{5} \quad d = \sqrt{74} \)  
8. \( m = \frac{13}{8} \quad d = \sqrt{233} \)  
9. \( m = \frac{41}{44} \quad d = \sqrt{3617} \)
We should first review some vocabulary and notation.

An equation is called a FUNCTION if there exists no more than one output for each input. If an equation has two or more outputs for a single input value, it is not a function. The set of possible inputs of an equation is called the DOMAIN, while the set of all possible outputs of an equation is called the RANGE.

Functions are often given names, most commonly “f,” “g” or “h.” The notation \( f(x) \) represents the output of a function, named \( f \), when \( x \) is the input. It is pronounced “\( f \) of \( x \).” The notation \( g(2) \), pronounced “\( g \) of 2,” represents the output of function \( g \) when \( x = 2 \).

Similarly, the function \( y = 3x + 4 \) and \( f(x) = 3x + 4 \) represent the same function. Notice that this notation is interchangeable, that is, \( y = f(x) \). In some textbooks, \( 3x + 4 \) is called the RULE of the function. The graph of \( f(x) = 3x + 4 \) is a line extending forever in both the \( x \) and \( y \) directions so the domain and range of \( f(x) \) are both all real numbers.

For each function below, tell the domain and range. Then find \( f(2) \) and solve \( f(x) = 3 \).

\[
f(x) = |x - 1| - 2
\]

Since you can use any real number for \( x \) in this equation, the domain is all real numbers. The smallest possible result for \( y \) is -2, so the range is \( y \geq -2 \). By looking at the graph or substituting \( x = 2 \) into the equation, \( f(2) = |2 - 1| - 2 = -1 \).

To solve \( f(x) = 3 \), find the points where the horizontal line \( y = 3 \) intersects the graph or solve the equation.

\[
3 = |x - 1| - 2
\]

\[
x = 4 \text{ or } -6.
\]

\[
f(x) \text{ is an unknown equation.}
\]

Any real number can replace \( x \), so the domain is all reals. The y-values are between \(-2 \) to \(+2 \) so the range is \(-2 \leq y \leq 2 \). By inspection \( f(2) = 1.8 \).

Since \(-2 \leq y \leq 2 \), \( f(x) = 3 \) has no solution.
You can only use numbers \(-3\) or larger for \(x\)-values so the domain is \(x \geq 3\). The smallest possible \(y\)-value is zero. The range is \(y \geq 0\).

Looking at the graph gives an approximate answer when \(x = 2\), 
\[ y \approx 2.25 \] or substituting \(x = 2\) into the equation, 
\[ f(2) = \sqrt[3]{x^2} = \sqrt[3]{4} \]
To solve \(f(x) = 3\), find the point where \(y = 3\) intersects the graph or solve
\[ 3 = \sqrt{x + 3} \]
\[ x = 6 \]

Now we can go back and try the original problem.

Given \(g(x) = 2(x + 3)^2\). State the domain and range. The graph is a parabola opening upward with locator point \((-3, 0)\). The domain is all real numbers and the range is \(y \geq 0\).

\[ g(-5) = 2(-5 + 3)^2 = 2(-2)^2 = 8 \]
\[ g(a + 1) = 2(a + 1 + 3)^2 = 2(a + 4)^2 = 2(a^2 + 8a + 16) = 2a^2 + 16a + 32 \]
\[ c) \quad \text{If} \quad g(x) = 32, \quad \text{then} \]
\[ 32 = 2(x + 3)^2 \]
\[ 16 = (x + 3)^2 \]
\[ \pm 4 = x + 3 \]
\[ x = 1 \text{ or } -7 \]
\[ d) \quad \text{If} \quad g(x) = 0, \quad \text{then} \]
\[ 0 = 2(x + 3)^2 \]
\[ 0 = (x + 3)^2 \]
\[ 0 = x + 3 \]
\[ x = -3 \]

Here are some more to try.

For each graph, tell the domain and range

1. 
2. 
3. 

4. If \(f(x) = 3 - x^2\), find \(f(5)\); find \(f(3a)\)
5. If \(g(x) = 5 - 3x^2\), find \(g(-2)\); find \(g(a+2)\)
6. If \(f(x) = \frac{x^1 + 3}{2x^1 + 15}\), find \(f(2)\); find \(f(2.5)\)
7. If \(f(x) = x^2 + 5x + 6\), solve \(f(x) = 0\)
8. If \(g(x) = 3(x - 5)^2\), solve \(g(x) = 27\)
9. If \(f(x) = (x + 2)^2\), solve \(f(x) = 27\)
Milepost Number 10

LS-53

Writing the Equation of a Line Given Two Points

The equation of a line is \( y = mx + b \) where \( m \) represents the slope and \( b \) represents the y-intercept. One way to find the equation is to calculate the slope and then solve for the y-intercept. A second method is to use the two points to write two equations involving \( m \) and \( b \) and then solve the system. Here is an example of each method.

Find an equation of the line passing through \((6, 5)\) and \((9, 2)\).

**Method One**

Using the points we make a generic slope triangle and find the slope to be:

\[ m = -\frac{3}{3} = -1. \]

The equation now looks like \( y = -1x + b \).

We can use either one of the two original points here; I will use \((9, 2)\).

Substitute the \( x \) and \( y \) values into the equation to give:

\[ 2 = -1(9) + b \]

which we can solve for \( b \):

\[ 2 = -9 + b \]

\[ 11 = b \]

Therefore the equation of the line is \( y = -1x + 11 \).

**Method Two**

Substitute the 2 points in for \( x \) and \( y \) in the equation \( y = mx + b \):

\[ 5 = 6m + b \]

\[ 2 = 9m + b \]

Subtracting the second equation from the first gives:

\[ 3 = -3m \quad \text{so} \quad m = -1. \]

We can now find \( b \) by substituting \( m = -1 \) into either equation. Using the first equation:

\[ 5 = 6(-1) + b \]

which we can solve for \( b \):

\[ 5 = -6 + b \]

\[ 11 = b \]

Therefore the equation of the line is \( y = -1x + 11 \).
Now we can go back and try the original problem. Write an equation for each line defined below.

a) The line through points (-1, 4) and (2, 1).

b) The line through points (6, 3) and (5, 5).

Using the first method for part (a). For the points we make a generic slope triangle and find the slope to be 

\[ m = \frac{3}{3} = -1 \]. The equation now looks like 

\[ y = -1x + b \].

Using the second method for part (b). We can use either one of the two original points here; I will use (-1, 4).

\[ 4 = -1(-1) + b \]
\[ 4 = 1 + b \]
\[ 3 = b \]

Therefore the equation of the line is 
\[ y = -1x + 3 \].

Here are some more to try. Find an equation of the line through the given points.

1. (2, 3) and (1, 2)
2. (-3, -5) and (-1, 0)
3. (4, 2) and (8, -1)
4. (1, 3) and (5, 7)
5. (0, 4) and (-1, -5)
6. (-3, 2) and (2, -3)
7. (4, 2) and (-1, -2)
8. (3, 1) and (-2, -4)
9. (4, 1) and (4, 10)

Answers:

1. \[ y = x + 1 \]
2. \[ y = \frac{5}{2} x + \frac{5}{2} \]
3. \[ y = -\frac{3}{4} x + 5 \]
4. \[ y = x + 2 \]
5. \[ y = 9x + 4 \]
6. \[ y = -x - 1 \]
7. \[ y = \frac{4}{5} x - \frac{6}{5} \]
8. \[ y = x - 2 \]
9. \[ x = 4 \]
Milepost Number 11
LS-110
Slopes of Parallel and Perpendicular Lines

Parallel lines have the same slopes. For perpendicular lines, the product of the slopes equal negative one. Here are two examples of problems involving parallel and perpendicular lines.

Find the equation of the line that is parallel to \( y = \frac{1}{2} x - 5 \) and passes through the point \((4, 10)\).

Any line parallel to the line \( y = \frac{1}{2} x - 5 \) which has slope \( \frac{1}{2} \), must also have slope \( \frac{1}{2} \). The equation must look like \( y = \frac{1}{2} x + b \). Substituting the given point in place of \( x \) and \( y \) we have:

\[
10 = \frac{1}{2}(4) + b
\]

Solving we find that \( b = 8 \) so that the equation is \( y = \frac{1}{2} x + 8 \).

Find the equation of the line that is perpendicular to \( y = \frac{1}{2} x - 5 \) and passes through the point \((4, 10)\).

Since the original line has slope \( \frac{1}{2} \), a perpendicular line must have slope \(-2\). The equation must look like \( y = -2x + b \). Substituting the given point in place of \( x \) and \( y \) we have:

\[
10 = -2(4) + b
\]

Solving we find that \( b = 18 \) so that the equation is \( y = -2x + 18 \).

We will now go back and solve the original problem.

Find an equation for each of the lines described below.

The line with slope \( \frac{1}{3} \) through the point \((0, 5)\).

The slope and the \( y \)-intercept of the line are both given so the equation is:

\[
y = \frac{1}{3} x + 5.
\]

The line parallel to \( y = 2x - 5 \) through the point \((1, 7)\).

The slope must be 2 so the equation must look like \( y = 2x + b \). Substituting the point for \( x \) and \( y \) we have:

\[
7 = 2(1) + b
\]

Solving we find that \( b = 5 \) so that the equation is \( y = 2x + 5 \).
The line perpendicular to \( y = 2x - 5 \) through the point (1, 7).

The slope must be \( -\frac{1}{2} \) so the equation must look like \( y = -\frac{1}{2} x + b \).

Substituting the coordinates of the point for \( x \) and \( y \) we have:

\[ 7 = -\frac{1}{2}(1) + b. \]

Solving we find that \( b = 7\frac{1}{2} \) so that the equation is \( y = -\frac{1}{2} x + 7\frac{1}{2} \).

The line through the point (0,0) so that the tangent of the angle it makes with the \( x \)-axis is 2.

The tangent of the angle is the same as the slope. The \( y \)-intercept is given so the equation must be:

\[ y = 2x \]

Here are some more to try. Find an equation for each of the lines described below.

1. The line with slope \( \frac{1}{3} \) passing through (-2, 5)
2. The line through the point (3, 0) so that the tangent of the angle it makes with the \( x \)-axis is -2.
3. The line parallel to \( y = \frac{2}{3} x + 5 \) passing through (3, 2)
4. The line perpendicular to \( y = \frac{2}{3} x + 5 \) passing through (3, 2).
5. The line parallel to \( 3x + 4y = 4 \) passing through (-4, 2)
6. The line perpendicular to \( 3x + 4y = 4 \) passing through (-4, 2).
7. The line parallel to the line determined by (-3, -2) and (2, 4) passing through (0, -1)
8. The line perpendicular to \( 2x - 3y = 6 \) passing through (0, 3).
9. The line parallel to \( y = 7 \) passing through (-2, 5)
10. The line perpendicular to \( y = 7 \) passing through (-2, 5).

Answers:

1. \( y = \frac{1}{3} x + 5 \frac{2}{3} \)
2. \( y = -2x + 6 \frac{3}{3} \)
3. \( y = \frac{2}{3} x \)
4. \( y = -\frac{3}{2} x + 6 \frac{1}{2} \)
5. \( y = -\frac{3}{4} x + 1 \)
6. \( y = \frac{4}{3} x + \frac{22}{3} \)
7. \( y = \frac{6}{5} x - 1 \)
8. \( y = -\frac{3}{2} x + 3 \)
9. \( y = 5 \)
10. \( x = -2 \)
Graphing Linear Inequalities

Graphing inequalities is very similar to graphing equations. First you graph the line. With inequalities you also need to determine if the line is solid (included) or dashed (not included) and which side of the line to shade. Here are two examples together in a system of inequalities.

On graph paper, graph and shade the solution for each of the systems of inequalities below. Describe each resulting region.

\begin{align*}
y &\leq \frac{2}{5} x \\
y &> 5 - x
\end{align*}

First, we treat the inequality as an equation: \( y = \frac{2}{5} x \). We can graph it using the slope, \( \frac{2}{5} \), and the y-intercept, \((0, 0)\). This line divides the grid into two regions. Choose a point in either region and check whether or not it makes the original inequality true. If the point \((0, 1)\) is the test point.

\[
1 \leq \frac{2}{5} (0) \quad \text{FALSE!}
\]

Since points above the line make the inequality false, points below the line must make it true. Therefore we shade all the points below the line to represent the solution.

Next we do the same thing for the second inequality. First treat it as an equation and graph it on the same set of axes. The slope is -1 and the y-intercept is \((0, 5)\). This line is dashed (not solid) because the inequality is strictly greater than, not greater than or equal to. The line has divided the grid into two regions and we will chose a point on one side as a test point. You can use the same point as last time. (This is not necessary and may not always be feasible. Here it is a convenient point.)

\[
1 > 5 - 0 \quad \text{FALSE!}
\]

Since our test point made the inequality false, the opposite side would make the inequality true. For the second inequality we shade above the line and to the right.

Putting these two inequalities together with their overlapping shading gives us the solution to the system of inequalities. In this case the solution is the darkest region, with a solid line above the region and a dashed line bordering below (left).
Now we can go back to the original problem. For this system of inequalities:

\[ \begin{align*}
  y &\leq -2x + 3 \\
  y &\geq x \\
  x &\geq -1
\end{align*} \]

a) Draw the graph. 

b) Find the area of the shaded region.

Start by looking at the equation of the line that marks the edge of each inequality. The first has slope -2 and y-intercept (0, 3). Checking (0, 1) gives a true statement so we shade below the solid line. The second has slope 1 and y-intercept (0, 0). Again checking (0, 1) gives a true statement so we shade above the solid line. The third is a vertical line at x = -1. Checking the point tells us to shade the right side. The overlapping shading is a triangle with vertices (-1, 5), (1, 1), and (-1, -1). The shaded area = \( \frac{1}{2} (6)(2) \).

Here are a few more to try. Graph and shade the solution for the system of inequalities below.

1. \[ \begin{align*}
  y &\leq -x + 2 \\
  y &\leq 3x - 6
\end{align*} \]

2. \[ \begin{align*}
  y &> \frac{2}{3} x + 4 \\
  y &< \frac{7}{12} x + 5
\end{align*} \]

3. \[ \begin{align*}
  x &< 3 \\
  y &\geq -2
\end{align*} \]

4. \[ \begin{align*}
  y &\leq 4x + 16 \\
  y &> \frac{4}{3} x - 4
\end{align*} \]

**Answers:**

1. 

2. 

3. 

4. 

Skill Building Resources
Multiplication and Division of Rational Expressions

To multiply or divide rational expressions you follow the same procedures as you did with numerical fractions. However, you need to first factor in order to simplify. Here are two examples.

**Problem A:** Multiply \( \frac{x^2 + 6x}{(x + 2)^2} \cdot \frac{x^2 + 7x + 6}{x^2 - 1} \) and simplify your result.

After factoring, our expression becomes:

\[
\frac{x(x + 6)}{(x + 2)(x + 6)} \cdot \frac{(x + 6)(x + 1)}{(x + 2)(x - 1)}
\]

After multiplying, reorder the factors:

\[
\frac{(x + 1)(x + 6)}{(x + 2)(x - 1)} \cdot \frac{x}{x + 2}
\]

Since \( \frac{(x + 6)}{(x + 2)} = 1 \) and \( \frac{(x + 1)}{(x + 1)} = 1 \), simplify:

\[
1 \cdot 1 \cdot \frac{x}{x + 2} \cdot 1 = \frac{x}{x + 2}
\]

**Problem B:** Divide \( \frac{x^2 - 4x - 5}{x^2 - 4x + 4} \div \frac{x^2 - 2x + 15}{x^2 + 4x + 12} \) and simplify your result.

First, change to a multiplication expression:

\[
\frac{x^2 - 4x - 5}{x^2 - 4x + 4} \cdot \frac{x^2 - 2x + 15}{x^2 + 4x + 12}
\]

After factoring, we get:

\[
\frac{(x - 5)(x + 1)}{(x - 2)(x - 2)} \cdot \frac{(x + 1)(x + 6)}{(x - 2)(x - 2)}
\]

Reorder the factors:

\[
\frac{(x - 5)}{(x - 5)} \cdot \frac{(x + 1)}{(x + 2)} \cdot \frac{(x + 2)}{(x + 6)} \cdot \frac{(x - 2)(x - 2)}{(x - 2)(x - 2)}
\]

Since \( \frac{(x - 5)}{(x - 5)} = 1 \) and \( \frac{(x + 2)}{(x + 2)} = 1 \), simplify:

\[
\frac{(x + 1)(x - 2)}{(x - 2)(x + 6)} = \frac{(x + 1)(x - 6)}{x^2 - x + 6}
\]

Thus, \( \frac{x^2 - 4x - 5}{x^2 - 4x + 4} \div \frac{x^2 - 2x + 15}{x^2 + 4x + 12} = \frac{(x + 6)(x - 6)}{x^2 - x + 6} \) or \( \frac{x^2 - 7x + 6}{x^2 + x - 6} \).

Now we can go back and try the original problem.

Think about factoring first, then perform each operation. Simplify when it is helpful.

**a)** \( \frac{x^2 + 5x + 6}{x^2 - 4x} \cdot \frac{4x}{x + 2} = \frac{(x + 2)(x + 3)}{x(x - 4)} \cdot \frac{4x}{x + 2} = \frac{(x + 3)}{x(x - 4)} \)

**c)** \( \frac{x^2 - 2x}{x^2 - 4x + 4} \cdot \frac{4x^2}{x - 2} = \frac{x(x - 2)}{(x - 2)(x - 2)} \cdot \frac{4x}{x} = \frac{1}{x} \)
Here are some more to try. Perform each operation.

1. \[
\frac{x^2 \div 16}{x \div 4} \cdot \frac{x^2 \div 3x \div 18}{x \div 2 \div 24}
\]

2. \[
\frac{x^2 \div x \div 6}{x \div 3x \div 10} \cdot \frac{x^2 \div x \div 15}{x \div 6x \div 9} \cdot \frac{x^2 \div 4x \div 21}{x^2 \div 9x \div 14}
\]

3. \[
\frac{x^2 \div x \div 6}{x^2 \div x \div 20} \cdot \frac{x^2 \div 6x \div 8}{x \div x \div 6}
\]

4. \[
\frac{x^2 \div x \div 30}{x^2 \div 13x \div 40} \cdot \frac{x^2 + 11x + 24}{x^2 \div 9x \div 18}
\]

5. \[
\frac{15 ! - 5x}{x^2 ! \div x ! \div ! 6} \div \frac{5x}{x^2 ! \div 6x ! \div ! 18}
\]

6. \[
\frac{17x ! + 119}{x^2 ! \div 15x ! \div ! 14} \div \frac{9x ! \div ! 1}{x^2 ! \div 13x ! \div ! 12}
\]

7. \[
\frac{2x^2 ! \div 15x ! \div ! 13}{3x^2 ! \div 10x ! \div ! 13} \cdot \frac{9x^2 ! \div ! 11}{4x^2 ! \div 4x ! \div ! 11}
\]

8. \[
\frac{x^2 ! \div ! 1}{x^2 ! \div 16x ! \div ! 17} + \frac{x^3 ! \div ! 12x}{x ! \div ! 17}
\]

Answers:

1. \((x+3)/(x-4)\)  
2. \((x-3)/(x-2)\)  
3. \((x+2)/(x-5)\)  
4. \((x+3)/(x-3)\)  
5. \(-(x + 4)/x\)  
6. \(17(x - 1)/(9x - 1)\)  
7. \((3x+1)/(2x+1)\)  
8. \(1/x(x+2)\)
To solve rational equations (equations with fractions) usually it is best to multiply everything by the common denominator to remove the fractions. This process is called FRACTION BUSTERS. Then solve the equation in the usual ways. Here are two examples.

\[
\frac{24}{x+1} = \frac{16}{1}
\]

Multiply both sides by the common denominator \((x + 1)\)

\[
(x + 1)\left(\frac{24}{x+1}\right) = (x + 1)\left(\frac{16}{1}\right)
\]

Then simplify.

\[
\begin{align*}
24 &= 16(x + 1) \\
24 &= 16x + 16 \\
8 &= 16x \\
\frac{8}{16} &= \frac{16x}{16} \\
x &= \frac{1}{2}
\end{align*}
\]

\[
\frac{5}{2x} + \frac{1}{6} = 8
\]

Multiply each term by the common denominator \(6x\).

\[
6x\left(\frac{5}{2x} + \frac{1}{6}\right) = 6x(8)
\]

Then simplify.

\[
\begin{align*}
6x\left(\frac{5}{2x}\right) + 6x\left(\frac{1}{6}\right) &= 48x \\
15 + x &= 48x \\
15 &= 47x \\
x &= \frac{15}{47}
\end{align*}
\]

Now we can go back any try the original problem. Solve each of the following rational equations.

a) \[
\frac{x}{3} = \frac{4}{x}
\]

\[
(3x)\left(\frac{x}{3}\right) = (3x)\left(\frac{4}{x}\right)
\]

\[
x^2 = 12
\]

\[
x = \pm\sqrt{12} = \pm2\sqrt{3}
\]

b) \[
\frac{x}{x+1} = \frac{4}{x}
\]

\[
x(x-1)\left(\frac{x}{x+1}\right) = x(x-1)\left(\frac{4}{x}\right)
\]

\[
x^2 = 4(x - 1)
\]

\[
x^2 - 4x + 4 = 0
\]

\[
(x - 2)(x - 2) = 0
\]

\[
x = 2
\]
c)
\[
\frac{1}{x} + \frac{1}{3x} = 6
\]
\[
3x\left(\frac{1}{x} + \frac{1}{3x}\right) = 3x(6)
\]
\[
3x\left(\frac{1}{x}\right) + 3x\left(\frac{1}{3x}\right) = 18x
\]
\[
\frac{3}{4} + 1 = 18x
\]
\[
x = \frac{2}{9}
\]

d)
\[
\frac{1}{x} + \frac{1}{x+1} = 3
\]
\[
x(x+1)\left(\frac{1}{x} + \frac{1}{x+1}\right) = x(x+1)3
\]
\[
x(x+1)\left(\frac{1}{x}\right) + x(x+1)\left(\frac{1}{x+1}\right) = x(x+1)3
\]
\[
x + 1 + x = 3x^2 + 3x
\]
\[
0 = 3x^2 + x - 1
\]
\[
\text{using the quadratic formula}
\]
\[
x \approx -0.43, -0.77
\]

Here are some more to try. Solve each of the following rational equations.

1. \[
\frac{3x}{5} = \frac{x + 12}{4}
\]
2. \[
\frac{4x + 11}{x} = 3x
\]
3. \[
\frac{2x}{5} - \frac{1}{3} = \frac{137}{3}
\]
4. \[
\frac{4x + 11}{x + 1} = x - 1
\]
5. \[
\frac{x}{3} = x + 4
\]
6. \[
\frac{x + 1 + 6}{3} = x
\]
7. \[
\frac{x + 1 + 6}{3} = x
\]
8. \[
\frac{2x + 13}{6} + \frac{1}{2} = \frac{x}{2}
\]
9. \[
\frac{3}{x} + \frac{5}{x + 1} = -2
\]
10. \[
\frac{2x + 13}{4} - \frac{x + 17}{6} = \frac{2x + 13}{12}
\]

Answers:
1. -10/7  2. 1/3, 1  3. 115 4. 0, 4  5. -6
6. ± 4 7. 3  8. 6  9. \[
\frac{3 \pm \sqrt{51}}{2}
\]
10. -13
Addition and subtraction of rational expressions is done the same way as addition and subtraction of numerical fractions. You change to a common denominator (if necessary), combine the numerators, and then simplify. Here is an example:

The Least Common Multiple (lowest common denominator) of 

\((x + 3)(x + 2)\) and \((x + 2)\) is 

\((x + 3)(x + 2)\).

The denominator of the first fraction already is the Least Common Multiple.

To get a common denominator in the second fraction, multiply the fraction

by \(\frac{x + 3}{x + 3}\), a form of one (1).

Multiply the numerator and denominator of the second term:

\[
\frac{4}{(x + 12)(x + 13)} + \frac{2x}{x + 12} \cdot \frac{x + 13}{x + 13}
\]

Distribute the numerator.

\[
\frac{4}{(x + 12)(x + 13)} + \frac{2x(x + 13)}{(x + 12)(x + 13)}
\]

Add, factor, and simplify.

\[
\frac{2x^2 + 16x + 14}{(x + 12)(x + 13)} = \frac{2(x + 14)}{(x + 12)(x + 13)}
\]

Now we can go back and try the original problem. Add or subtract and simplify.

\[a) \quad \frac{2x - 1x}{x + 14} + \frac{3x + 16}{x + 14} = \frac{2x - 1x + 13x - 16}{x + 14} \]

\[b) \quad \frac{3}{(x + 12)(x + 13)} + \frac{x}{(x + 12)(x + 13)} = \frac{3 + 1x}{(x + 12)(x + 13)} \]

\[= \frac{3 + 1x}{(x + 12)(x + 13)} \]

\[= \frac{x}{x + 12} \]
Here are some more to try. Add or subtract and simplify.

1. \( \frac{x}{(x+12)(x+13)} + \frac{2}{(x+12)(x+13)} \)

2. \( \frac{8x+13}{2x+13} - \frac{2x+16}{2x+13} \)

3. \( \frac{6}{x(x+15)} + \frac{2}{x+13} \)

4. \( \frac{3x+11}{x^2+16} - \frac{3x+15}{x^2+18x+16} \)

5. \( \frac{7x+1}{x^2+12x+13} - \frac{6x}{x^2+12x+12} \)

6. \( \frac{3}{x+1} + \frac{4}{1-x} + \frac{1}{x} \)

7. \( \frac{3y}{9y^2+14x^2} - \frac{1}{3y+12x} \)

8. \( \frac{2}{x+14} - \frac{x+14}{x^2+16} \)

9. \( \frac{5x+19}{x^2+12x+13} + \frac{6}{x^2+17x+112} \)

10. \( \frac{x+14}{x^2+13x+12} + \frac{x+15}{x^2+12x+13} \)

Answers:

1. \( \frac{1}{x+3} \)

2. \( \frac{3(2x+3)}{2x+3} = 3 \)

3. \( \frac{2x^2+19}{x(x+13)(x+13)} \)

4. \( \frac{4(5x+186)}{(x+14)(x+14)^2} \)

5. \( \frac{x+12}{(x+13)(x+12)} \)

6. \( \frac{-1}{x(x+11)} \)

7. \( \frac{2x}{(3y+12x)(3y+12x)} \)

8. \( \frac{1}{x+14} \)

9. \( \frac{5(x+12)}{(x+14)(x+11)} \)

10. \( \frac{2x}{(x+17)(x+7)} \)
There are some basic rules for integral and rational exponents. The patterns are summarized below with some examples:

\[ x^0 = 1. \quad \text{Examples:} \quad 2^0 = 1, \quad (-3)^0 = 1, \quad \left(\frac{1}{4}\right)^0 = 1. \]

\[ x^{-n} = \frac{1}{x^n}. \quad \text{Examples:} \quad x^{-3} = \frac{1}{x^3}, \quad y^{-4} = \frac{1}{y^4}, \quad 4^{-2} = \frac{1}{4^2} = \frac{1}{16}. \]

\[ \frac{1}{x^{-n}} = x^n. \quad \text{Examples:} \quad \frac{1}{x^5} = x^{-5}, \quad \frac{1}{x^{-2}} = x^2, \quad \frac{1}{\sqrt{3}} = 3^2 = 9. \]

\[ x^{\frac{a}{b}} = \left(x^a\right)^{\frac{1}{b}} = \sqrt[b]{x^a} \quad \text{Examples:} \quad 5^2 = \sqrt[2]{5}, \quad 16^\frac{3}{2} = (\sqrt{16})^3 = 2^3 = 8 \]

\[ 4^\frac{2}{3} = \sqrt[3]{4^2} = \sqrt[3]{16} = 2\sqrt[3]{2}. \]

We can now go back and try the original problem. Use integral or rational exponents to write each of the following as a power of \( x \).

a) \( \sqrt[5]{x} = x^{1/5} \) (using the fourth property above)

b) \( \frac{1}{x^3} = x^{-3} \) (using the second property above)

c) \( \sqrt[3]{x^2} = x^{2/3} \) (using the fourth property above)

d) \( \frac{1}{\sqrt{x^2}} = \frac{1}{x^{1/2}} = x^{-1/2} \) (using properties four and two above)
Here are some more to try. Use integral or rational exponents to simplify each expression. You should not need a calculator for any of these.

1. \(x^{-5}\)  
2. \(m^0\)  
3. \(4^{-1}\)  
4. \(y^3\)  
5. \(5^{-2}\)  
6. \(5^0\)  
7. \(y^{-7}\)  
8. \((x^3y^4)^{-2}\)  
9. \(x^{-1}y^{-8}\)  
10. \(x^{-4}y^{-2}(x^{-3}y^{-6})^0\)  
11. \(25^{1/2}\)  
12. \(25^{-1/2}\)  
13. \(21^{1/2}\)  
14. \(\sqrt[3]{\frac{1}{27}}\)  
15. \(x^{3/2}\)  
16. \(9^{3/2}\)  
17. \((x^3y^6)^{1/3}\)  
18. \(16^{-3/4}\)  
19. \((m^2)^{-3/2}\)  
20. \((x^3y^6)^{1/2}\)  
21. \((9x^3y^6)^{-2}\)

Answers:

1. \(\frac{1}{x^5}\)  
2. 1  
3. \(\frac{1}{4}\)  
4. \(\frac{1}{y^3}\)  
5. \(\frac{1}{x^2}\)  
6. 1  
7. \(\frac{1}{y^7}\)  
8. \(\frac{1}{x^6y^8}\)  
9. \(\frac{1}{xy^8}\)  
10. \(\frac{1}{x^4y^2}\)  
11. 5  
12. \(\frac{1}{5}\)  
13. \(\sqrt{2}\)  
14. 3  
15. \(\sqrt{x^3} = x\sqrt{x}\)  
16. 27  
17. \(xy^2\)  
18. \(\frac{1}{8}\)  
19. \(\frac{1}{m^3}\)  
20. \(xy^3\sqrt{x}\)  
21. \(\frac{1}{81x^6y^{12}}\)
Completing the Square and Locator Point for a Parabola

If a parabola is in graphing form then the locator point or vertex is easily found and a sketch of the graph can quickly be made. If the equation of the parabola is not in graphing form the equation needs to be rearranged. One way to rearrange the equation is by completing the square. Here are three examples:

Note that first two examples are in your book problem PG-46 shown with algebra tiles. You might like that way better because you can "see it" or the way shown below which just gives a rule.

First recall that \( y = x^2 \) is the parent equation for parabolas and then the graphing equation for that function is given by

\[
y = a(x - h)^2 + k
\]

where \((h, k)\) is the locator point, and relative to the parent graph the function has been:
vertically stretched, if the absolute value of \(a\) is greater than 1
vertically compressed, if the absolute value of \(a\) is less than 1
reflected across the x-axis, if \(a\) is less than 0.

**Example 1:** \( y = x^2 + 8x + 10 \)

We need to make \( x^2 + 8x \) into a perfect square. Taking half of the \(x\) coefficient and squaring it will accomplish the task.

\[
\begin{align*}
\text{The 16 that was put into the parenthesis must be compensated for by subtracting 16.} \\
y &= (x^2 + 8x + 16) + 10 - 16 \\
&= (x + 4)^2 - 6. \text{ The locator is } (-4, -6).
\end{align*}
\]

**Example 2:** \( y = x^2 + 5x + 2 \)

We need to make \( x^2 + 5x \) into a perfect square. Again, taking half of the \(x\) coefficient and squaring it will always accomplish the task.

\[
\begin{align*}
\text{The } \frac{25}{4} \text{ that was put into the parenthesis must be compensated for by subtracting } \frac{25}{4}. \\
y &= (x^2 + 5x + \frac{25}{4}) + 2 - \frac{25}{4} \\
&= (x + \frac{5}{2})^2 - 4 \frac{1}{4}. \text{ The locator is } (-\frac{5}{2}, -4 \frac{1}{4})
\end{align*}
\]
Example 3: \( y = 2x^2 - 6x + 2 \)

This problem is a little different because we have \(2x^2\). First we must factor the 2 out of the x-terms. Then we make \(x^2 - 3x\) into a perfect square as before.

The \( \frac{9}{4} \) that was put into the parenthesis must be compensated for by subtracting 
\[ 2\left( \frac{9}{4} \right) = \frac{9}{2}. \]

Factor and simplify

We can now go back to the original problem. Write the equation in graphing form and sketch a graph of \( y = 2x^2 - 4x + 5 \).

First we must factor the 2 out of the x-terms. Then we make \(x^2 - 2x\) into a perfect square as before.

The 1 that was put into the parenthesis must be compensated for by subtracting 
\[ 2(1) = 2 \]

Factor and simplify

You might also do this problem by completing two perfect squares using the algebra tiles. You will have three extra ones, giving the +3 on the end of the graphing form.

Here are some more to try. Write the equation in graphing form. Tell the locator and stretch factor.

1. \( y = x^2 - 6x + 9 \)  
2. \( y = x^2 + 3 \)  
3. \( y = x^2 - 4x \)  
4. \( y = x^2 + 2x - 3 \)  
5. \( y = x^2 + 5x + 1 \)  
6. \( y = x^2 - \frac{1}{3}x \)  
7. \( y = 3x^2 - 6x + 1 \)  
8. \( y = 5x^2 + 20x - 16 \)  
9. \( y = -x^2 - 6x + 10 \)

Answers:

1. (3, 0) normal size  
2. (0, 3) normal  
3. (2, -4) normal  
4. (-1, -4) normal  
5. \((-\frac{5}{2}, -\frac{5}{4})\) normal  
6. \((\frac{1}{6}, -\frac{1}{36})\) normal  
7. (1, -2) stretch by 3  
8. (-2, -36) stretch by 5  
9. (-3, 19) normal size but upside down
Absolute value means the distance from a reference point. There is a pattern used to solve absolute
value equations and two patterns used for the different inequalities. They are shown below and then
some examples are solved.

**Examples**

\[ |x| = k \text{ means: } x = k \text{ or } x = -k \]

\[ |x| = 5 \text{ means } x = 5 \text{ or } x = -5 \]
\hspace{1cm} (5 or -5 are 5 units from zero)

\[ |x| < k \text{ means: } -k < x < k \]

\[ |x| < 5 \text{ means } -5 < x < 5 \]
\hspace{1cm} (the numbers between -5 and 5 are less than 5 units from zero)

\[ |x| > k \text{ means: } x > k \text{ or } x < -k \]

\[ |x| > 5 \text{ means } x > 5 \text{ or } x < -5 \]
\hspace{1cm} (the numbers greater than 5 or less than -5 are more than 5 units from zero)

If the expression inside the absolute value is more complicated, you still follow one of the three basic
patterns above. Also \( \leq \) and \( \geq \) use the same patterns as the pure inequality.

If \( |2x + 3| = 7 \), then the quantity \((2x + 3)\) must equal \( 7 \) or \(-7\).

\[
\begin{align*}
2x + 3 &= 7 & \text{or} & & 2x + 3 &= -7 \\
2x &= 4 & \text{or} & & 2x &= -10 \\
x &= 2 & \text{or} & & x &= -5
\end{align*}
\]

If \( |2x + 3| \leq 7 \), then the quantity \((2x + 3)\) must be between \(-7\) and \(7\).

\[
\begin{align*}
-7 &\leq 2x + 3 \leq 7 \\
-10 &\leq 2x \leq 4 \\
-5 &\leq x \leq 2
\end{align*}
\]
Now we can go back and try the original problem. Solve each absolute value equation or inequality.

\[ 2|2x + 3| = 10 \]

(isolate the absolute value)
\[ |2x + 3| = 5 \]

(using the first pattern)
\[ 2x + 3 = 5 \quad \text{or} \quad 2x + 3 = -5 \]
\[ 2x = 2 \quad \text{or} \quad 2x = -8 \]
\[ x = 1 \quad \text{or} \quad x = -4 \]

\[-|x + 3| < 10 \]

(isolate the absolute value)
\[ |x + 3| > 10 \]

(dividing by a negative changes the sign) Since any absolute value is never negative, the solution is all numbers.

Here are some more to try. Solve each absolute value equation or inequality.

1. \( |x - 2| + 10 = 8 \)
2. \( 15 - |x + 1| = 3 \)
3. \( -3 \cdot |x + 6| + 12 = 0 \)
4. \( |2x + 7| = 0 \)
5. \( |x + 4| \geq 7 \)
6. \( |x| - 5 \leq 8 \)
7. \( |4r - 2| > 8 \)
8. \( -2|x - 3| + 6 < -4 \)
9. \( |4d| \leq 7 \)

Answers:
1. No solution
2. \( x = 11, -13 \)
3. \( x = -2, -10 \)
4. \( x = -\frac{7}{2} \)
5. \( x \geq 3 \text{ or } x \leq -11 \)
6. \( -13 \leq x \leq 13 \)
7. \( r < -\frac{3}{2} \text{ or } r \geq \frac{5}{2} \)
8. \( x > 8 \text{ or } x < -2 \)
9. \( -3 \leq d \leq 11 \)

Skill Building Resources
Exponential functions are equations of the form \( y = km^x \) where \( k \) represents the initial value, \( m \) represents the multiplier, \( x \) represents the time. Some problems just involve substituting in the information and doing the calculations. If you are trying to solve for the time \( (x) \), then you will usually need to use logarithms. If you need to find the multiplier \( (m) \), then you will need roots. Here are some examples.

Lunch at our favorite fast food stand now cost $6.50. The price has steadily increased 4% per year for many years.

What will lunch cost in 10 years?  
The initial value is $6.50, the multiplier is 1.04, and the time is 10 years. Substituting into the formula:  
\[
y = 6.50(1.04)^{10} = 9.62
\]

What did it cost 10 years ago?  
\[
y = 6.50(1.04)^{-10} = 4.39
\]

How long before lunch costs $10?  
The initial value is $6.50, the multiplier is 1.04, and the time is unknown but the final value is $10. Substituting into the formula:  
\[
10 = 6.50(1.04)^x
\]

This time we must solve an equation.  
\[
(1.04)^x = \frac{10}{6.50} = 1.538
\]

\[
x = \frac{\log(1.538)}{\log(1.04)} \approx 11 \text{ years}
\]

Tickets for the big concert first went on sale three weeks ago for $60. This week people are charging $100.

What was the weekly multiplier and weekly percent increase?  
The initial value is $60, the time is 3 weeks, and the final value is $100. Substituting into the formula:  
\[
100 = 60k^3
\]

\[
k^3 = \frac{100}{60} = 1.667
\]

\[
k = \sqrt[3]{1.667} \approx 1.186
\]

The multiplier was about 1.186 so it was a weekly increase of about 18.6%.
We can now go back and solve the original problem parts (b), (c), and (d).

When rabbits were first brought to Australia, they had no natural enemies. There were about 80,000 rabbits in 1866. Two years later, in 1868, the population had grown to over 2,400,000!

b) Write an exponential equation for the number of rabbits \( t \) years after 1866. For 1866, 80,000 would be the initial value, time would be 2 years, and the final amount would be 2,400,000. Here is the equation to solve:

\[
2,400,000 = 80,000m^2
\]

so the multiplier \( m = \sqrt{30} \approx 5.477 \).

The desired equation is: \( R = 80,000(5.477)^t \)

c) How many rabbits do you predict would have been present in 1871? The initial value is still 80,000, the multiplier \( \approx 5.477 \) and now the time is 5 years.

\[
80,000(5.477)^5 \approx 394 \text{ million}
\]

d) According to your model, in what year was the first pair of rabbits introduced into Australia? Now 2 is the initial value, 80,000 is the final value, the multiplier is still 5.477 but the time is not known. Here is the equation to solve:

\[
80,000 = 2(5.477)^x
\]

\[
40,000 = (5.477)^x
\]

\[
x = \frac{\log(40000)}{\log(5.477)} \approx 6.23 \text{ years, so some time during 1859.}
\]

Here are some more to try.

1. A video tape loses 60% of its value every year it is in the store. The video cost $80 new. Write a function that represents its value in \( t \) years. What is it worth after 4 years?

2. Inflation is at a rate of 7% per year. Janelle's favorite bread now costs $1.79. What did it cost 10 years ago? How long before the cost of the bread doubles?

3. Find the initial value if five years from now, a bond that appreciates 4% per year will be worth $146.

4. Sixty years ago when Sam's grandfather was a kid he could buy his friend dinner for $1.50. If that same dinner now costs $25.25 and inflation was consistent, write an equation that will give you the costs at different times.

5. A two-bedroom house in Omaha is now worth $110,000. If it appreciates at a rate of 2.5% per year, how long will it take to be worth $200,000?

6. A car valued at $14,000 depreciates 18% per year. After how many years will the value have depreciated to $1000?

Answers:

1. \( y = 80(0.4)^t \), $2.05  
2. $91, 10.2 years  
3. $120  
4. \( y = 1.50(1.048)^x \)  
5. 24.2 years  
6. 13.3 years
Finding the Equation for the Inverse of a Function

To find the equation for the inverse of a function just interchange the $x$ and $y$ variables and then solve for $y$. This also means that the coordinates of points that are on the graph of the function will be reversed on the inverse. Here are some examples:

If $y = 2(x + 3)$ then the inverse is: $x = 2(y + 3)$.

Solving for $y$ to get the final answer:

$$(y + 3) = \frac{x}{2}$$
$$y = \frac{x}{2} - 3$$

If $y = \frac{1}{2}(x + 4)^2 + 1$ the inverse is: $x = \frac{1}{2}(y + 4)^2 + 1$.

Solving for $y$ to get the final answer:

$$\frac{1}{2}(y + 4)^2 = x - 1$$
$$(y + 4)^2 = 2x - 2$$
$$y + 4 = \pm \sqrt{2x - 2}$$
$$y = \pm \sqrt{2x - 2} - 4$$

Note that because of the $\pm$, this inverse is not a function.

If $y = -\frac{2}{3}x + 6$ then the inverse is: $x = -\frac{2}{3}y + 6$.

Solving for $y$ to get the final answer:

$$-\frac{2}{3}y = x - 6$$
$$y = -\frac{3}{2}(x - 6) = -\frac{3}{2}x + 9$$

If $y = \sqrt{x^2 - 12} + 5$ then the inverse is: $x = \sqrt{y^2 - 12} + 5$.

Solving for $y$ to get the final answer:

$$\sqrt{y^2 - 12} = x - 5$$
$$y^2 - 2 = (x - 5)^2$$
$$y = (x - 5)^2 + 2$$

Note that since the original function is one half of a parabola, the graph of the inverse function is also only one half of a parabola.
We can now go back and try the original problem:

Find the equation for the inverse of the following function: \( y = 2\sqrt{3(x - 1)} + 5 \). Sketch the graph of both the original and the inverse.

Interchanging \( x \) and \( y \) we get \( x = 2\sqrt{3(y - 1)} + 5 \). Solving for \( y \) to get the final answer:

\[
2\sqrt{3(y - 1)} = x - 5 \\
\sqrt{3(y - 1)} = \frac{x - 5}{2} \\
3(y - 1) = \frac{(x - 5)^2}{4} \\
y - 1 = \frac{(x - 5)^2}{12} \\
y = \frac{(x - 5)^2}{12} + 1
\]

For the original function:
Domain: \( x \geq 1 \); Range: \( y \geq 5 \)

Some points on the original graph are:
(1, 5), (\( \frac{7}{3} \), 9), (4, 11)--half a parabola.

For the inverse function:
Domain: \( x \geq 5 \); Range: \( y \geq 1 \).

Some points on the inverse graph are:
(5, 1), (\( \frac{7}{3} \)), (11, 4)--half a parabola.

Here are some more to try. Find the equation for the inverse of each function.

1. \( y = 3x + 2 \) 
2. \( y = \frac{x + 1}{4} \) 
3. \( y = \frac{1}{3}x + 2 \)
4. \( y = x^3 + 1 \) 
5. \( y = 1 + \sqrt{x+5} \) 
6. \( y = 3(x+2)^2 \) 
7. \( y = 2\sqrt{x-1} + 3 \) 
8. \( y = \frac{-1}{2+x} \) 
9. \( y = \log_3(x^2) \)

Answers:

1. \( y = \frac{x + 2}{3} \) 
2. \( y = 4x + 1 \) 
3. \( y = 3x + 6 \)
4. \( y = 2\sqrt{x+1} \) 
5. \( y = (x+1)^2 \) 
6. \( y = \sqrt{\frac{x + 7}{3}} - 2 \)
7. \( y = \sqrt{\frac{x+3}{2}} + 1 \) 
8. \( y = \frac{1}{x^2} \) 
9. \( y = 3x + 2 \)
Solving a System of Equations in Three Variables

To solve a system of equations in three variables using elimination you use the same basic process as you do with two variables only you have to do it twice. Choose any variable to eliminate and then you are left with two equations in two variables. Continue to solve in the usual way. To solve using matrix multiplication you need to change the system into matrices and then isolate the variable matrix by using the inverse matrix on the graphing calculator. Here is an example of each method.

Solve for \((x, y, z)\)

\[
\begin{align*}
5x - 4y - 6z &= -19 \\
-2x + 2y + z &= 5 \\
3x - 6y - 5z &= -16
\end{align*}
\]

Method One--Elimination

Choose a variable to eliminate. Any is possible. We choose \(z\). Use the first two equations. Multiply the second equation by 6 and add it to the first.

\[
\begin{align*}
6(-2x + 2y + z &= 5) &= -12x + 12y + 6z = 30 \\
(\text{**}) \\
5x - 4y - 6z &= -19 \\
-7x + 8y &= 11
\end{align*}
\]

Then we must also eliminate \(z\) using two other equations. Multiply the second by 5 and add it to the third.

\[
\begin{align*}
5(-2x + 2y + z &= 5) &= -10x + 10y + 5z = 25 \\
(\text{***}) \\
3x - 6y - 5z &= -16 \\
-7x + 4y &= 9
\end{align*}
\]

Now we have two equations with two variables. Using lines (**) and (***) we can subtract the second line from the first to eliminate \(x\) and find \(y\).

\[
\begin{align*}
(\text{**}) - (\text{***}) &= -7x + 8y = 11 \\
(\text{***}) - (\text{**) &= -7x + 4y = 9 \\
y &= \frac{1}{2}
\end{align*}
\]

Using our answer for \(y\) in (**) we can find \(x\).

\[
\begin{align*}
-7x + 8\left(\frac{1}{2}\right) &= 11 \\
-7x &= 7 \\
x &= -1
\end{align*}
\]

Finally go back to any of the original equations to find \(z\). Using the first one:

\[
\begin{align*}
5(-1) - 4\left(\frac{1}{2}\right) - 6z &= -19 \\
-7 - 6z &= -19 \\
-6z &= -12 \\
z &= 2
\end{align*}
\]

The solution is \((-1, \frac{1}{2}, 2)\)
Method Two—Matrices

Write the problem as a matrix equation.

\[
\begin{bmatrix} 5 & 4 & 6 \\ 2 & 1 & -1 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 5 \\ 16 \end{bmatrix}
\]

Left multiplying both sides of the equation by the inverse of the coefficient matrix gives:

\[
\begin{bmatrix} 5 & 4 & 6 \\ 2 & 1 & -1 \\ 3 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 & 6 \\ 2 & 1 & -1 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 4 & 6 \\ 2 & 1 & -1 \\ 3 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 19 \\ 5 \\ 16 \end{bmatrix}
\]

(You may need to refer to you calculator directions or your resource page from unit 5 for help with entering this in your graphing calculator.

The solution is (-1, \( \frac{1}{2} \), 2)

We can now go back and solve the original question.

Use elimination or matrix multiplication to solve this system of equations:

\[
\begin{align*}
x + y - z &= 12 \\
3x + 2y + z &= 6 \\
2x + 5y - z &= 10
\end{align*}
\]

**Method One**

Adding equations one and two eliminates z.

\[
\begin{align*}
x + y - z &= 12 \\
3x + 2y + z &= 6 \\
4x + 3y &= 18 \quad (**)
\end{align*}
\]

Adding equations two and three also eliminates z.

\[
\begin{align*}
3x + 2y + z &= 6 \\
2x + 5y - z &= 10 \\
5x + 7y &= 16 \quad (***)
\end{align*}
\]

Now we have two equations in two variables. Multiplying (***) by 5 and (***) by -4 eliminates x.

\[
\begin{align*}
5(4x + 3y) &= 20x + 15y = 90 \\
-4(5x + 7y) &= -20x - 28y = -64 \\
-13y &= 26 \Rightarrow y = -2
\end{align*}
\]

Using \( y = -2 \) into (***) gives x = 6.

Using \( y = -2 \) and \( x = 6 \) in any of the original equations gives z = -8

The solution is (6, -2, -8).

**Method Two**

Write the system in matrices.

\[
\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}
\]

Isolate the variable matrix.

\[
\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}
\]

Use the graphing calculator to multiply.

\[
\begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}
\]

\[
\begin{align*}
x &= 6 \\
y &= -2 \\
z &= -8
\end{align*}
\]

The solution is (6, -2, -8).
Here are some more to try. Use elimination or matrix multiplication to solve these system of equations. Most teachers expect their students to be able to use both methods successfully.

1. \(x + y + z = 34\)
   \(3x + 2y + 4z = 95\)
   \(x + 2y + 3z = 56\)
2. \(x - 2y + 3z = 8\)
   \(2x + y + z = 6\)
   \(x + y + 2z = 12\)
3. \(5x + y + 2z = 6\)
   \(3x - 6y - 9z = -48\)
   \(x - 2y + z = 12\)
4. \(4x - y + z = -5\)
   \(2x + 2y + 3z = 10\)
   \(5x - 2y + 6z = 1\)
5. \(x + y = 2 - z\)
   \(-y + 1 = -z - 2x\)
   \(3x - 2y + 5z = 16\)
6. \(a - b + 2c = 2\)
   \(a + 2b - c = 1\)
   \(2a + b + c = 4\)
7. \(-4x = z - 2y + 12\)
   \(y + z = 12 - x\)
   \(8x - 3y + 4z = 1\)
8. \(3x + y - 2z = 6\)
   \(x + 2y + z = 7\)
   \(6x + 2y - 4z = 12\)
9. \(4x + 4y - 5z = -2\)
   \(2x - 4y + 10z = 6\)
   \(x + 2y + 5z = 0\)

**Answers:**
1. \((17, 12, 5)\)
2. \((-1, 3, 5)\)
3. \((-1, -3, 7)\)
4. \((-1, 3, 2)\)
5. \((-3, 0, 5)\)
6. no solution
7. \((-3, 5, 10)\)
8. infinite solutions
9. \(\left(\frac{1}{2}, -\frac{3}{4}, \frac{1}{5}\right)\)