The Difference Quotient

In these exercises, we practice computing and simplifying an expression known as the difference quotient. At first, we will work with simpler building blocks to the difference quotient.

1. This exercise is just a warm-up. There is no difference quotient here. Let \( f(x) = 3x^2 + 2x - 4 \).

   a) Find \( f(0) \).
   \[
   = 3(0)^2 + 2(0) - 4 = -4
   
   b) Find \( f(1) \).
   \[
   = 3(1)^2 + 2(1) - 4 = 3 + 2 - 4 = 1
   
   c) Simplify \( f(-x) \).
   \[
   = 3(-x)^2 + 2(-x) - 4 = 3x^2 - 2x - 4
   
   d) Simplify \( -f(x) \).
   \[
   = - (3x^2 + 2x - 4) = -3x^2 - 2x + 4
   
   e) Simplify \( f(x + 1) \).
   \[
   = 3(x+1)^2 + 2(x+1) - 4
   = 3(x^2 + 2x + 1) + 2x + 2 - 4
   = 3x^2 + 6x + 3 + 2x - 2
   = 3x^2 + 8x + 1
   
   f) Simplify \( f(2x) \).
   \[
   = 3(2x)^2 + 2(2x) - 4 = 3(4x^2) + 4x - 4
   = 12x^2 + 4x - 4
   
2. If \( g(x) = x^2 + 3 \), simplify \( g(x + 2) - g(x) \).
   \[
   = (x+2)^2 + 3 - (x^2 + 3)
   = x^2 + 4x + 4 + 3 - x^2 - 3
   = 4x + 4
   
3. If \( k(x) = \frac{3}{x} \), simplify \( k(x + 2) - k(x) \).
   \[
   = \frac{3}{x+2} - \frac{3}{x}
   = \frac{3x}{x(x+2)} - \frac{3(x+2)}{x(x+2)}
   = \frac{3x}{x(x+2)} - \frac{3x + 6}{x(x+2)}
   = \frac{-6}{x(x+2)}
4. For each of these functions, find and simplify the difference quotient. For a function $f$, the difference quotient of $f$ is $\frac{f(x+h) - f(x)}{h}$.

a) $f(x) = 4x + 3$

$$\frac{f(x+h) - f(x)}{h} = \frac{4(x+h) + 3 - (4x+3)}{h} = \frac{4x + 4h + 3 - 4x - 3}{h} = \frac{4h}{h} = 4, \quad h \neq 0$$

b) $g(x) = x^2 - 4$

$$\frac{g(x+h) - g(x)}{h} = \frac{(x+h)^2 - 4 - (x^2 - 4)}{h} = \frac{x^2 + 2xh + h^2 - 4 - x^2 - 4}{h} = \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h, \quad h \neq 0$$

c) $p(x) = 2x^2 - 3x + 1$

$$\frac{p(x+h) - p(x)}{h} = \frac{2(x+h)^2 - 3(x+h) + 1 - (2x^2 - 3x + 1)}{h} = \frac{2(x^2 + 2xh + h^2) - 3x - 3h + 1 - 2x^2 + 3x - 1}{h} = \frac{2x^2 + 4xh + 2h^2 - 3h - 2x^2}{h} = \frac{4xh + 2h^2 - 3h}{h} = \frac{h(4x + 2h - 3)}{h} = 4x + 2h - 3, \quad h \neq 0$$

d) $k(x) = \frac{1}{x+3}$

$$\frac{k(x+h) - k(x)}{h} = \frac{1}{x+h+3} - \frac{1}{x+3} = \frac{1}{x+h+3} - \frac{1}{x+3} = \frac{1}{h} \left( \frac{x+3}{x+3} \right) - \frac{x+h+3}{x+3} \cdot \frac{x+3}{x+h+3} = \frac{x+3 - (x+h+3)}{h(x+h+3)(x+3)} = \frac{-h}{h(x+h+3)(x+3)} = \frac{-1}{(x+h+3)(x+3)}, \quad h \neq 0$$

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