Math 261 Lecture Notes: Section 4.1

Vector Spaces and Subspaces

Definition 1. A vector space is a nonempty set $V$ of objects, called vectors, on which two operations, called addition and scalar multiplication are defined and for which the following ten axioms hold for all $\vec{u}, \vec{v}$ and $\vec{w}$ in $V$ and for all scalars $c$ and $d$:

(1) The sum of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} + \vec{v}$ is in $V$ (closure under addition)
(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity of addition)
(3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity of addition)
(4) There is a zero vector, $\vec{0}$, in $V$ such that $\vec{v} + \vec{0} = \vec{v}$ (existence of an additive identity)
(5) For each $\vec{u}$ in $V$, there is a vector $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$ (existence of an additive inverse)
(6) The scalar multiple of $\vec{u}$ by $c$, denoted by $c\vec{u}$, is in $V$ (closure under scalar multiplication)
(7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
(8) $(c + d)\vec{u} = c\vec{u} + c\vec{v}$
(9) $c(d\vec{u}) = cd\vec{u}$
(10) $1\vec{u} = \vec{u}$

We have already seen many examples of vector spaces:

- The set of vectors in $\mathbb{R}^n$ under the standard operations of vector addition and scalar multiplication is a vector space over the set of real numbers.

- The set of all real $m \times n$ matrices under the standard operations of vector addition and scalar multiplication is a vector space over the set of all real numbers.
**Definition 2.** A set is a collection of objects. For sets $A$ and $B$, we say $B$ is a subset of $A$ if every element of $B$ is contained in $A$. Below are some common sets of numbers you have seen:

<table>
<thead>
<tr>
<th>Set</th>
<th>Set Notation</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complex Numbers</td>
<td>${a + bi \mid a, b \in \mathbb{R}}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>Real Numbers</td>
<td>${a \in \mathbb{R}}$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>Irrational Numbers</td>
<td>${a \in \mathbb{R} \mid a \notin \mathbb{Q}}$</td>
<td>$\mathbb{Q}^c$</td>
</tr>
<tr>
<td>Rational Numbers</td>
<td>${\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0}$</td>
<td>$\mathbb{Q}$</td>
</tr>
<tr>
<td>Integers</td>
<td>${\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>Whole Numbers</td>
<td>${0, 1, 2, 3, \ldots}$</td>
<td>$\mathbb{N} \cup {0}$</td>
</tr>
<tr>
<td>Natural Numbers</td>
<td>${1, 2, 3, \ldots}$</td>
<td>$\mathbb{N}$</td>
</tr>
<tr>
<td>Binary Numbers</td>
<td>${0, 1}$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

**Examples of Subsets.**

- The real numbers is a subset of the set of complex numbers. This is true because every real number can be written as $a + (0)i$.

- The set of integers is a subset of the real numbers. The set of integers is a subset of the rational numbers.

- The set $A = \{1, 3, 5\}$ is a subset of the set $B = \{1, 2, 4, 5\}$.

Figure 1: Diagram of Subsets of Numbers
Showing Closure Under Addition:

Consider the subset

\[ S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \]

of

\[ V = \mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\} . \]

Define addition and scalar multiplication in the standard way for vectors.

The set \( S \) is \textit{closed under addition} because when two arbitrary vectors from this set are added, the resultant vector is an element of \( S \). To show this, we write the following:

Let \( \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \) and let \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \) be vectors in \( S \). Then

\[ \vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ 0 \end{bmatrix} , \]

which is an element of \( S \). Therefore \( S \) is closed under addition.

Showing Closure Under Addition Does Not Hold:

Consider the subset

\[ S = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \mid x \in \mathbb{R} \right\} \]

of

\[ V = \mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\} . \]

Define addition and scalar multiplication with the standard componentwise operations.

Let \( \vec{u} = \begin{bmatrix} u_1 \\ 1 \end{bmatrix} \) and let \( \vec{v} = \begin{bmatrix} v_1 \\ 1 \end{bmatrix} \) be vectors in \( S \). Then

\[ \vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 2 \end{bmatrix} , \]

which is not an element of \( S \). Therefore the \( S \) is not closed under addition.
Example 1. Determine if the previous two sets are closed under scalar multiplication.

Definition 3. A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has the following three properties:

1. The zero vector of $V$ is in $H$.
2. $H$ is closed under vector addition. (That is, for every $\vec{u}$ and $\vec{v}$ in $H$, it holds that $\vec{u} + \vec{v}$ is in $H$.)
3. $H$ is closed under scalar multiplication. (That is, for every $\vec{u}$ in $H$ and every scalar $c$, it holds that $c\vec{u}$ is in $H$.)
Example 2. Show that the set $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \bigg| y = 2x \right\}$ is a subspace of $\mathbb{R}^2$ with standard vector addition and scalar multiplication.

Theorem 1. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are in a vector space $V$, then $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a subspace of $V$. 
Example 3. Show that the set $V = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ does not form a vector under matrix addition and scalar multiplication.
Example 4. Is \( S \) a subspace of \( \mathbb{R}^3 \) when \( S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \)?

Example 5. Is \( S \) a subspace of \( \mathbb{R}^2 \) when \( S = \left\{ \begin{bmatrix} s \\ 3s - t \end{bmatrix} \mid s, t \in \mathbb{R} \right\} \)?
The vector space of all polynomials of degree $n$ or less is denoted by $P_n$ with standard addition and scalar multiplication. This set is assumed to contain 0, although this is not technically a “polynomial” by all definitions.

**Example 6.** Show that the set of polynomials $S = \{ax^2 + c \mid a, c \in \mathbb{R}\}$ is a subspace of $P_2$.

**Example 7.** Show that the set of polynomials $S = \{ax^2 + 1 \mid a \in \mathbb{R}\}$ is not a subspace of $P_2$. 