8.1 Power Functions

A power function is of the form $f(x) = a_n x^n$ where $a_n$ is a real number and $n$ is a non-negative integer.
8.1.1 Graphs of Basic Power Functions

Figure 8.1: $y = x^2$  
Figure 8.2: $y = x^3$  
Figure 8.3: $y = x^4$  
Figure 8.4: $y = x^5$

Notice how the graphs of $y = x^2$ and $y = x^4$ are very similar? This same behavioral pattern also holds for $y = x^6$, $y = x^8$, and so on. Similarly, the graphs of $y = x^3$ and $y = x^5$ are very similar. This behavioral pattern continues for $y = x^5$, $y = x^7$, and so on.

The key distinction between even and odd power functions is what happens in the long run. Notice how both ends of the graph of $y = x^2$ point upward? This holds true for all power functions with an even degree. For $f(x) = x^2$, we can summarize this behavior in the following way:

- As $x \to \infty$, $f(x) \to \infty$.
- As $x \to -\infty$, $f(x) \to \infty$.

Conversely, notice how one end of the graph of $y = x^3$ points upward and one end points downward? This holds for all power functions with an odd degree. For $f(x) = x^3$, we can summarize this behavior in the following way:

- As $x \to \infty$, $f(x) \to \infty$.
- As $x \to -\infty$, $f(x) \to -\infty$.

Figure 8.5: Even Powers  
Figure 8.6: Odd Powers
8.2 Polynomial Functions

A polynomial function is of the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are real numbers and \( n \) is a non-negative integer.

The degree of the polynomial is \( n \).

The leading term is \( a_n x^n \). This determines the long-run behavior of the function.

8.2.1 Graphical Properties of Polynomial Functions

**Figure 8.7**

- Degree: 2
- Max. # of zeros: 2
- Max. # of turning points: 1

**Figure 8.8**

- Degree: 3
- Max. # of zeros: 3
- Max. # of turning points: 2

**Figure 8.9**

- Degree: 4
- Max. # of zeros: 4
- Max. # of turning points: 3

**Figure 8.10**

- Degree: 5
- Max. # of zeros: 5
- Max. # of turning points: 4
8.2.2 Determining the Multiplicity of Zeros

A polynomial function \( f \) has a real zero \( r \) if and only if \( (x - r) \) is a factor of \( f(x) \).

If \( r \) is a zero of even multiplicity, then the factor \( (x - r) \) occurs an even number of times. The graph then looks like the graph of an even power function at that zero. Hence the function “bounces” there.

If \( r \) is a zero of odd multiplicity, then the factor \( (x - r) \) occurs an odd number of times. The graph then looks like the graph of an odd power function at that zero. Hence, if \( (x - r) \) occurs once, the function passes “straight through” at that zero and if \( (x - r) \) occurs any other odd number of time, the function “flattens” there.

Example 1: Let \( f(x) = 4x(x - 7)^2(x + 1)^5(x + 2)^3 \). Determine the zeros and their respective multiplicities. Also state the degree of the function and the long-run behavior.

The zeros of the function are 0, 7, -1, and -2. Table 8.1 below shows how the factors correspond to each zero and determine the multiplicity:

<table>
<thead>
<tr>
<th>factor</th>
<th>zero</th>
<th>multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( (x - 7)^2 )</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>( (x + 1)^5 )</td>
<td>-1</td>
<td>5</td>
</tr>
<tr>
<td>( (x + 2)^3 )</td>
<td>-5</td>
<td>3</td>
</tr>
</tbody>
</table>

The degree of the function is 11. The long run behavior for this function is that it behaves like the function \( y = 4x^{11} \). More specifically, as \( x \to \infty, f(x) \to \infty \) and as \( x \to -\infty, f(x) \to -\infty \).
8.2.3 Graphing Polynomial Functions

**Example 2:** Graph the polynomial function defined by \( f(x) = -\frac{1}{2}(x - 2)(x + 4) \) by finding the following: the degree of the polynomial, the long run behavior, the maximum number of turning points, the horizontal and vertical intercepts, and the zeros and their multiplicity.

- In the long run, the function \( f \) behaves like the function defined by \( g(x) = -\frac{1}{2}x^2 \), which is obtained by finding the leading term. Using limit notation, we write:
  
  As \( x \to \infty \), \( f(x) \to -\infty \)
  
  As \( x \to -\infty \), \( f(x) \to -\infty \)

- The degree of \( f \) is 2.

- The maximum number of turning points is 1.

- The horizontal intercepts occur where \( f(x) = 0 \):
  
  \[ 0 = -\frac{1}{2}(x - 2)(x + 4) \]
  
  \[ x - 2 = 0 \quad \text{or} \quad x + 4 = 0 \]
  
  \[ x = 2 \quad \text{or} \quad x = -4 \]

  The horizontal intercepts are (2, 0) and (-4, 0).

- The zeros of \( f \) are 2 and -4; each has a multiplicity of 1. Thus the graph goes “straight through” the \( x \)-axis at 2 and at -4.

- The vertical intercept occurs where \( x = 0 \):
  
  \[ f(0) = -\frac{1}{2}(0 - 2)(0 + 4) \]
  
  \[ = 4 \]

  The vertical intercept is (0, 4).

**We know that there will be one turning point in the graph of \( y = -\frac{1}{2}(x - 2)(x + 4) \) as there are two zeros each of multiplicity 1. With a parabola, the turning point is the vertex, which can be found to be \((-1, 3.5)\). For any polynomial with degree larger than 2, we will use technology OR estimate where the turning points occur and what the maximum/minimum values there are.
Example 3: Graph the polynomial function defined by \( f(x) = \frac{1}{4}(x + 1)^2(x + 2)(x - 5) \) by finding the following: the degree of the polynomial, the long run behavior, the maximum number of turning points, the horizontal and vertical intercepts, and the zeros and their multiplicities.

**Figure 8.13:** Hand-drawn graph (y scale omitted)

- In the long run, the function \( f \) behaves like the function defined by \( g(x) = \frac{1}{4}x^4 \), which is obtained by finding the leading term. Using limit notation, we write:

  \[
  \text{As } x \to \infty, \ f(x) \to \infty \\
  \text{As } x \to -\infty, \ f(x) \to \infty
  \]

- The degree of \( f \) is 4.

- The maximum number of turning points is 3.

- The horizontal intercepts occur where \( f(x) = 0 \): If \( f(x) = 0 \), then \( x = -1, \ x = -2, \) or \( x = 5 \). The horizontal intercepts are \((-1, 0), (-2, 0)\) and \((5, 0)\).

- The zeros of \( f \) are -2, -1, and 5.

<table>
<thead>
<tr>
<th>zeros</th>
<th>multiplicity</th>
<th>behavior at zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>&quot;straight through&quot;</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>&quot;bounces&quot; or &quot;touches&quot;</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>&quot;straight through&quot;</td>
</tr>
</tbody>
</table>

Your book is much more specific. You do not need to be!!!

- The vertical intercept occurs where \( x = 0 \):

  \[
  f(0) = \frac{1}{4}(0 + 1)^2(0 + 2)(0 - 5) \\
  = -2.5
  \]

  The vertical intercept is \((0, -2.5)\).

- The local minimum can be found using a calculator. We will *estimate* it as some negative number when drawing the graph without a calculator.

**Figure 8.14:** Calculator Graph
Example 4: Graph the polynomial function defined by \( f(x) = -\frac{1}{2}x(x + 3)(x - 2)^3 \) by finding the following: the degree of the polynomial, the long run behavior, the maximum number of turning points, the horizontal and vertical intercepts, and the zeros and their multiplicities.

- In the long run, the function \( f \) behaves like the function defined by \( g(x) = -\frac{1}{2}x^5 \), which is obtained from the leading term. Using limit notation, we write:
  
  As \( x \to \infty \), \( f(x) \to -\infty \)
  
  As \( x \to -\infty \), \( f(x) \to \infty \)

- The degree of \( f \) is 5.

- The maximum number of turning points is 4.

- The horizontal intercepts occur where \( f(x) = 0 \): If \( f(x) = 0 \), then \( x = 0 \), \( x = -3 \), or \( x = 2 \). The horizontal intercepts are \((0,0), (-3,0)\) and \((2,0)\).

- The zeros of \( f \) are -3, 0, and 2.

<table>
<thead>
<tr>
<th>zeros</th>
<th>multiplicity</th>
<th>behavior at zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>1</td>
<td>&quot;straight through&quot;</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>&quot;straight through&quot;</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>&quot;flattens&quot;</td>
</tr>
</tbody>
</table>

Your book does not distinguish between a zero of multiplicity 1 and a zero of multiplicity 3. You do not have to, but it is helpful for drawing a reasonable graph.

- The vertical intercept occurs where \( x = 0 \):

  \[
  f(0) = -\frac{1}{2}(0)(0 + 3)(0 - 2)^3 \\
  = 0
  \]

  The vertical intercept is \((0,0)\).

- The local minimum and local maximums can be found using a calculator. When drawing the graph without a calculator, this point is estimated.
8.2.4 Finding the Formula of a Polynomial Function from Its Graph

Example 5: Find a possible formula for the polynomial function graphed in Figure 8.16. Clearly state the zeros and their multiplicities.

- The zeros of $f$ are -2, 0, and 3.

<table>
<thead>
<tr>
<th>zero</th>
<th>multiplicity</th>
<th>factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>$(x - (-2))$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$(x - 0)$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$(x - 3)$</td>
</tr>
</tbody>
</table>

- A possible formula is of the form:

$$f(x) = k(x + 2)(x)(x - 3)$$

- To find an appropriate value for $k$, we will use one ordered pair (other than a horizontal intercept). The graph contains $(1, -3)$, so $f(1) = -3$:

$$-3 = k(1 + 2)(1)(1 - 3)$$
$$-3 = k(-6)$$
$$\frac{1}{2} = k$$

- A possible formula for the function is:

$$f(x) = \frac{1}{2}x(x + 2)(x - 3)$$
Example 6: Find a possible formula for the polynomial function graphed in Figure 8.17. Clearly state the zeros and their multiplicities.

- The zeros of $f$ are -3, 0, and 3.

<table>
<thead>
<tr>
<th>zero</th>
<th>multiplicity</th>
<th>factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>even: 2</td>
<td>$(x + 3)^2$</td>
</tr>
<tr>
<td>0</td>
<td>even: 2</td>
<td>$x^2$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$(x - 3)$</td>
</tr>
</tbody>
</table>

- A possible formula is of the form:

$$f(x) = kx^2(x + 3)^2(x - 3)$$

- To find an appropriate value for $k$, we will use one ordered pair (other than a horizontal intercept). The graph contains $(1, 2)$, so $f(1) = 2$:

$$2 = k(1)^2(1 + 3)^2(1 - 3)$$
$$2 = k(-32)$$
$$-\frac{1}{16} = k$$

- A possible formula for the function is:

$$f(x) = -\frac{1}{16}x^2(x + 3)^2(x - 3)$$

- It is good to verify that this formula matches the maximum degree of the function and the leading coefficient. The long-run behavior of the function shows that the leading coefficient should be negative (it is!) and the function should have an odd degree (it does!).
Example 7: Find a possible formula for the polynomial function graphed in Figure 8.18. Clearly state the zeros and their multiplicities.

- The zeros of $f$ are -1 and 2.

<table>
<thead>
<tr>
<th>zero</th>
<th>multiplicity</th>
<th>factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>odd: 3</td>
<td>$(x + 1)^3$</td>
</tr>
<tr>
<td>2</td>
<td>even: 2</td>
<td>$(x - 2)^2$</td>
</tr>
</tbody>
</table>

- A possible formula is of the form:

$$f(x) = k(x + 1)^3(x - 2)^2$$

- To find an appropriate value for $k$, we will use one ordered pair (other than a horizontal intercept). The graph contains $(0, -4)$, so $f(0) = -4$:

$$-4 = k(0 + 1)^3(0 - 2)^2$$
$$-4 = k(4)$$
$$-1 = k$$

- A possible formula for the function is:

$$f(x) = -(x + 1)^3(x - 2)^2$$

- It is good to verify that this formula matches the maximum degree of the function and the leading coefficient. The long-run behavior of the function shows that the leading coefficient should be negative (it is!) and the function should have an odd degree (it does!).

Figure 8.18
8.3 Properties of Rational Functions

A rational function is of the form \( R(x) = \frac{p(x)}{q(x)} \) where \( p \) and \( q \) are polynomial functions.

The zeros of a rational function occur where \( p(x) = 0 \), as the function’s value is zero where the value of the numerator is zero.

A rational function is undefined where \( q(x) = 0 \), as the function is undefined whenever its denominator is zero.

8.3.1 Basic Rational Functions

Figure 8.19: Graph of \( y = \frac{1}{x} \)

Table 8.2

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \frac{1}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-( \frac{1}{4} )</td>
</tr>
<tr>
<td>-2</td>
<td>-( \frac{1}{2} )</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-( \frac{1}{2} )</td>
<td>-2</td>
</tr>
<tr>
<td>-( \frac{1}{4} )</td>
<td>-4</td>
</tr>
<tr>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>4</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

The horizontal asymptote is \( y = 0 \). We can summarize the behavior related to the horizontal asymptote with the following two statements:

- As \( x \to \infty \), \( f(x) \to 0 \).
- As \( x \to -\infty \), \( f(x) \to 0 \).

The vertical asymptote is \( x = 0 \).
The horizontal asymptote is \( y = 0 \) and the vertical asymptote is \( x = 0 \). We can summarize the behavior related to the horizontal asymptote with the following two statements:

- As \( x \to \infty \), \( f(x) \to 0 \).
- As \( x \to -\infty \), \( f(x) \to 0 \).

### 8.3.2 Basic Rational Functions (Close Up)

#### Table 8.3

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \frac{1}{x^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>( \frac{1}{16} )</td>
</tr>
<tr>
<td>-2</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>-( \frac{1}{2} )</td>
<td>4</td>
</tr>
<tr>
<td>-( \frac{1}{4} )</td>
<td>16</td>
</tr>
<tr>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>16</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{16} )</td>
</tr>
</tbody>
</table>

#### Figure 8.21: Even Powers

#### Figure 8.22: Odd Powers
8.3.3 Rational Function Transformations

Example 8: Use transformations to graph the function defined by \( R(x) = \frac{1}{x - 1} \). Identify any horizontal asymptotes, any vertical asymptotes, the domain, and the range.

The graph of \( y = \frac{1}{x - 1} \) is the graph of \( y = \frac{1}{x} \) shifted right 1 unit.

The horizontal asymptote is \( y = 0 \) and the vertical asymptote is \( x = 1 \).

The domain of \( R \) is \((-\infty, 1) \cup (1, \infty)\). The range of \( R \) is \((-\infty, 0) \cup (0, \infty)\).

Example 9: Use transformations to graph the function defined by \( R(x) = -\frac{1}{x + 2} \). Identify any horizontal asymptotes, any vertical asymptotes, the domain, and the range.

The graph of \( y = -\frac{1}{x + 2} \) is the graph of \( y = \frac{1}{x} \) reflected across the \( x \)-axis and then shifted left 2 units.

The horizontal asymptote is \( y = 0 \) and the vertical asymptote is \( x = -2 \).

The domain of \( R \) is \((-\infty, -2) \cup (-2, \infty)\). The range of \( R \) is \((-\infty, 0) \cup (0, \infty)\).
Example 10: Use transformations to graph the function defined by \( R(x) = -\frac{3}{x^2} \). Identify any horizontal asymptotes, any vertical asymptotes, the domain, and the range.

The graph of \( y = -\frac{3}{x^2} \) is the graph of \( y = \frac{1}{x^2} \) reflected across the \( x \)-axis and stretched vertically by a factor of 3.

The horizontal asymptote is \( y = 0 \) and the vertical asymptote is \( x = 0 \).

The domain of \( R \) is \((-\infty, 0) \cup (0, \infty)\). The range of \( R \) is \((-\infty, 0) \cup (0, \infty)\).

Example 11: Use transformations to graph the function defined by \( R(x) = \frac{1}{(x-3)^2} - 2 \). Identify any horizontal asymptotes, any vertical asymptotes, the domain, and the range.

The graph of \( y = \frac{1}{(x-3)^2} - 2 \) is the graph of \( y = \frac{1}{x^2} \) shifted 3 units right and 2 units down.

The horizontal asymptote is \( y = -2 \) and the vertical asymptote is \( x = 3 \).

The domain of \( R \) is \((-\infty, 3) \cup (3, \infty)\). The range of \( R \) is \((-\infty, -2) \cup (-2, \infty)\).
8.3.4 Determining the Horizontal Asymptote of a Rational Function

Let \( m \) be the degree of the function \( p \) in the numerator and let \( n \) be the degree of the function \( q \) in the denominator.

- If \( m < n \), then the horizontal asymptote is \( y = 0 \).
- If \( m = n \), then the horizontal asymptote is \( y = c \) where \( c \) is a real number determined by the ratio of leading coefficients.
- If \( m > n \), then no horizontal asymptote exists.

If \( m = n + 1 \), then an \textit{oblique asymptote} exists.

Example 12: Determine any horizontal asymptotes for the following rational functions.

(a) \( R(x) = \frac{5x + 1}{10x^2 + 6x} \)
   Since the degree of the numerator is 1 and the degree of the denominator is 2, the horizontal asymptote is \( y = 0 \).

(b) \( R(x) = \frac{5x^2 + 1}{10x^2 + 6x} \)
   Since the degrees of the numerator and denominator are 2, the horizontal asymptote is \( y = \frac{5}{10} \), which simplifies to \( y = \frac{1}{2} \).

(c) \( R(x) = \frac{5x^3 + 1}{10x^2 + 6x} \)
   Since the degree of the numerator is 3 and the degree of the denominator is 2, no horizontal asymptote exists. An oblique asymptote exists.

(d) \( R(x) = \frac{5x^4 + 1}{10x^2 + 6x} \)
   Since the degree of the numerator is 4 and the degree of the denominator is 2, no horizontal or oblique asymptote exists.
8.3.5 Determining Zeros, Holes, and Vertical Asymptotes

The **zeros** of a rational function are the values of \( x \) for which \( p(x) = 0 \), as the function’s value is zero where the value of the numerator is zero. Most of the time, the zeros will occur at \( a \) when the factor \( (x - a) \) is in the numerator of \( R \).

A rational function is undefined where \( q(x) = 0 \), as this would cause division by zero.

A **vertical asymptote** occurs when the denominator of the simplified form of \( R \) is equal to zero. Most of the time, the vertical asymptote \( x = b \) will occur when the factor \( (x - b) \) is in the denominator of the simplified form of \( R \).

A **hole** occurs when **both** the numerator and denominator equal zero for some value of \( x \). We will identify a zero at \( c \) when the linear factor \( (x - c) \) occurs in both the numerator and denominator of a rational function. Note that during simplification this factor cancels and results in a domain restriction for \( R \).

**Example 13:** Find the any zeros, holes, vertical asymptotes, and horizontal asymptotes for each rational function below. Factor each expression first and reduce to lowest terms if necessary.

(a) \( R(x) = \frac{x - 5}{x + 6} \)

\[
R(x) = \frac{x - 5}{x + 6}
\]

The function \( R \) is already in lowest terms.

The function \( R \) has a zero of 5.

The vertical asymptote is \( x = -6 \).

The horizontal asymptote is \( y = 1 \) as the degree of the numerator and denominator is the same and the ratio of leading terms is \( \frac{x}{x} \).
(b) \( R(x) = \frac{-10x}{5x - 5} \)

\[
R(x) = \frac{-10x}{5x - 5} = \frac{-10x}{5(x - 1)} = \frac{-2x}{x - 1}
\]

The function \( R \) has a zero of 0.
The vertical asymptote is \( x = 1 \).
The horizontal asymptote is \( y = -2 \) as the degree of the numerator and denominator is the same and the ratio of leading terms is \( \frac{-2x}{x} \).

(c) \( R(x) = \frac{x^2 - 5x - 6}{x^2 + x - 12} \)

\[
R(x) = \frac{x^2 - 5x - 6}{x^2 + x - 12} = \frac{(x - 6)(x + 1)}{(x + 4)(x - 3)}
\]

The function \( R \) has zeros 6 and -1.
The vertical asymptotes are \( x = -4 \) and \( x = 3 \).
The horizontal asymptote is \( y = 1 \) as the degree of the numerator and denominator is the same and the ratio of leading terms is \( \frac{x^2}{x^2} \).

(d) \( R(x) = \frac{x^2 + 4x + 3}{2x^2 - 8} \)

\[
R(x) = \frac{x^2 + 4x + 3}{2x^2 - 8} = \frac{(x + 1)(x + 3)}{2(x^2 - 4)} = \frac{(x + 1)(x + 3)}{2(x - 2)(x + 2)}
\]

The function \( R \) has zeros \(-1\) and \(-3\).
The vertical asymptotes are \( x = 2 \) and \( x = -2 \).
The horizontal asymptote is \( y = \frac{1}{2} \) as the degree of the numerator and denominator is the same and the ratio of leading terms is \( \frac{x^2}{2x^2} \).
(e) \[ R(x) = \frac{8}{x^2 - 25} \]

\[ R(x) = \frac{8}{x^2 - 25} = \frac{8}{(x - 5)(x + 5)} \]

The function \( R \) does not have any zeros.

The vertical asymptotes are \( x = 5 \) and \( x = -5 \).

The horizontal asymptote is \( y = 0 \) as the degree of the numerator is less than the degree of the denominator.

(f) \[ R(x) = \frac{3x - 6}{x^2 + x - 6} \]

\[ R(x) = \frac{3x - 6}{x^2 + x - 6} = \frac{3(x - 2)}{(x + 3)(x - 2)} = \frac{3}{x + 3}, \ x \neq 2 \]

The function \( R \) does not have any zeros as the numerator (of the simplified \( R(x) \)) is not zero for any value of \( x \).

The function \( R \) has a hole when \( x = 2 \), which occurs at the point \( (2, \frac{3}{5}) \).

The vertical asymptote is \( x = -3 \).

The horizontal asymptote is \( y = 0 \) as the degree of the numerator is less than the degree of the denominator.
(g) \[ R(x) = \frac{3}{x^3 - 4x} \]

\[ R(x) = \frac{3}{x(x - 2)(x + 2)} \]

The function \( R \) does not have any zeros as the numerator is not zero for any value of \( x \).

The vertical asymptotes are \( x = 0, x = 2 \) and \( x = -2 \).

The horizontal asymptote is \( y = 0 \) as the degree of the numerator is less than the degree of the denominator.

(h) \[ R(x) = \frac{2(x - 1)(x + 7)^2}{(x - 1)(x + 3)(x + 4)} \]

\[ R(x) = \frac{2(x - 1)(x + 7)^2}{(x - 1)(x + 3)(x + 4)} \]

\[ = \frac{2(x + 7)^2}{(x + 3)(x + 4)}, \quad x \neq 1 \]

The function \( R \) has a zero of \(-7\).

The function \( R \) has a hole when \( x = 1 \). As \( \frac{2(1+7)^2}{(1+3)(1+4)} = \frac{32}{5} \), this occurs at the point \( (1, \frac{32}{5}) \).

The vertical asymptotes are \( x = -3 \) and \( x = -4 \).

The horizontal asymptote is \( y = 0 \) as the degree of the numerator is less than the degree of the denominator.