Definition 1: Given two functions $f$ and $g$, the composite function, denoted by $f \circ g$ (read “$f$ composed with $g$” or “$f$ of $g$”) is defined by

$$(f \circ g)(x) = f(g(x))$$

The function $g$ is referred to as the inside function and the function $f$ is referred to as the outside function.

In determining the domain for the composite function, the domain of the inside function and the domain for the resultant composite function must be accounted for.
4.1 Composing Functions Algebraically

Example 1: Let \( f(x) = 5x^2 + 3x - 1 \) and \( g(x) = 2x - 7 \). Compute the following:

(a) \( (f \circ g)(2) \)

\[
(f \circ g)(2) = f(g(2)) \\
= f(2(2) - 7) \\
= f(-3) \\
= 5(-3)^2 + 3(-3) - 1 \\
= 45 - 9 - 1 \\
= 35
\]

(b) \( (g \circ f)(0) \)

\[
(g \circ f)(0) = g(f(0)) \\
= g(5(0)^2 + 3(0) - 1) \\
= g(-1) \\
= 2(-1) - 7 \\
= -9
\]

(c) \( (f \circ f)(-1) \)

\[
(f \circ f)(-1) = f(f(-1)) \\
= f(5(-1)^2 + 3(-1) - 1) \\
= f(1) \\
= 5(1)^2 + 3(1) - 1 \\
= 7
\]
(d) \( (g \circ f)(x) \)

\[
(g \circ f)(x) = g(f(x)) \\
= g(5x^2 + 3x - 1) \\
= 2(5x^2 + 3x - 1) - 7 \\
= 10x^2 + 6x - 2 - 7 \\
= 10x^2 + 6x - 9
\]

(e) \( (f \circ g)(x) \)

\[
(f \circ g)(x) = f(g(x)) \\
= f(2x - 7) \\
= 5(2x - 7)^2 + 3(2x - 7) - 1 \\
= 5(4x^2 - 28x + 49) + 6x - 21 - 1 \\
= 20x^2 - 140x + 245 + 6x - 21 - 1 \\
= 20x^2 - 134x + 223
\]

(f) \( (g \circ g)(x) \)

\[
(g \circ g)(x) = g(g(x)) \\
= g(2x - 7) \\
= 2(2x - 7) - 7 \\
= 4x - 14 - 7 \\
= 4x - 21
\]
Example 2: Find \((g \circ f)(x)\) if \(f(x) = \frac{5}{x + 4}\) and \(g(x) = \frac{3x}{2x - 1}\). State the domain of \(g \circ f\).

\[
(g \circ f)(x) = g(f(x)) \\
= g\left(\frac{5}{x+4}\right) \\
= 3\left(\frac{5}{x+4}\right) \\
= \frac{3\left(\frac{5}{x+4}\right)}{2\left(\frac{5}{x+4}\right) - 1} \\
= \frac{3\left(\frac{5}{x+4}\right)}{2\left(\frac{5}{x+4}\right) - 1} \cdot \frac{x+4}{x+4} \\
= \frac{15}{2(5) - 1(x+4)} \\
= \frac{15}{10 - x - 4} \\
= \frac{15}{-x + 6} \\
= \frac{-15}{x - 6}
\]

The original expression shows that \(x \neq -4\). The fully simplified expression shows that \(x \neq 6\). Thus the domain of \(f \circ g\) is \(\{x \mid x \neq -4, 6\}\).
Example 3: Find \((f \circ g)(x)\) if \(f(x) = \frac{5}{x + 4}\) and \(g(x) = \frac{3x}{2x - 1}\). State the domain of \(f \circ g\).

\[
(f \circ g)(x) = f(g(x))
\]

\[
= f\left(\frac{3x}{2x - 1}\right)
\]

\[
= \frac{5}{\left(\frac{3x}{2x - 1}\right) + 4}
\]

\[
= \frac{5}{\left(\frac{3x}{2x - 1}\right) + 4} \cdot \frac{2x - 1}{2x - 1}
\]

\[
= \frac{5(2x - 1)}{3x + 4(2x - 1)}
\]

\[
= \frac{10x - 5}{3x + 8x - 4}
\]

\[
= \frac{10x - 5}{11x - 4}
\]

In the original expression, we see that \(2x - 1 \neq 0\) and thus \(x \neq \frac{1}{2}\). In the simplified expression, we see that \(11x - 4 \neq 0\) and thus \(x \neq \frac{4}{11}\). Therefore the domain of \(f \circ g\) is \(\{x \mid x \neq \frac{1}{2}, \frac{4}{11}\}\).
**Example 4:** Let \( j(x) = \frac{4}{x^2 + 1} \) and \( k(x) = \sqrt{2x - 3} \). Find \( (j \circ k)(x) \) and state the domain of \( j \circ k \).

\[
(j \circ k)(x) = j(k(x)) \\
= j(\sqrt{2x - 3}) \\
= \frac{4}{(\sqrt{2x - 3})^2 + 1} \\
= \frac{4}{2x - 3 + 1} \\
= \frac{4}{2x - 2} \\
= \frac{4}{2(x - 1)} \\
= \frac{2}{x - 1}
\]

In the original expression, \( 2x - 3 \geq 0 \) and thus \( x \geq \frac{3}{2} \). From the final, simplified expression, we see that \( x \neq 1 \). Noting that \( x \neq 1 \) when \( x \geq \frac{3}{2} \), we determine the domain of \( j \circ k \) to be \( \{x \mid x \geq \frac{3}{2}\} \). As an interval, we would write \( \left[\frac{3}{2}, \infty\right) \).
Example 5: Let $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}(x - 1)$. Show that both $(f \circ g)(x) = x$ and that $(g \circ f)(x) = x$ for every $x$ in the respective domains of $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x))$$

$$= f\left(\frac{1}{2}(x - 1)\right)$$

$$= 2\left(\frac{1}{2}(x - 1)\right) + 1$$

$$= (x - 1) + 1$$

$$= x$$

$$(g \circ f)(x) = g(f(x))$$

$$= g(2x + 1)$$

$$= \frac{1}{2}((2x + 1) - 1)$$

$$= \frac{1}{2}(2x)$$

$$= x$$
4.2 Composing Functions Numerically

Example 6: Use the functions $f$ and $g$ given in Table 4.1 to determine the following.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>5</td>
<td>4</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>9</td>
</tr>
</tbody>
</table>

(a) $g(f(2))$

To find $g(f(2))$, we will first find $f(2)$. After that, we will apply the function $g$:

$$g(f(2)) = g(1) = -2$$

(b) $f(g(2))$

To find $f(g(2))$, we will first find $g(2)$. Since $g(2)$ is 9 and 9 is not in the domain of $f$, we cannot apply the function $f$. Therefore $f(g(2))$ is undefined.

(c) $g(g(-1))$

To find $g(g(-1))$, we will first find $g(-1)$. After that, we will apply the function $g$ again:

$$g(g(-1)) = g(2) = 9$$

(d) $f(f(0))$

To find $f(f(0))$, we will first find $f(0)$. After that, we will apply the function $f$ again:

$$f(f(0)) = f(-2) = 5$$
4.3 Composing Functions Graphically

Example 7: Use Figure 4.1 to complete the following, if they exist.

![Figure 4.1](image)

(a) $f(g(2))$
   
   To find $f(g(2))$, we will first find $g(2)$. After that, we will apply the function $f$:
   
   $$f(g(2)) = f(-1)$$
   $$= -2$$

(b) $g(f(-3))$
   
   To find $g(f(-3))$, we will first find $f(-3)$. After that, we will apply the function $g$:
   
   $$g(f(-3)) = g(-2)$$
   $$= 7$$

(c) $g(g(0))$
   
   To find $g(g(0))$, we will first find $g(0)$. After that, we will apply the function $g$ again:
   
   $$g(g(0)) = g(3)$$
   $$= -3$$

(d) $f(f(1))$
   
   To find $f(f(1))$, we will first find $f(1)$. Since $f(1)$ is 6 and 6 is not in the domain of $f$, we cannot apply the function $f$. Therefore $f(f(1))$ is undefined.
Example 8: For the following examples, find the functions $f$ and $g$ such that $H = f \circ g$. Do not choose $f(x) = x$ or $g(x) = x$.

The easiest approach to decomposing functions is to think about how you would say them verbally. For example, $H(x) = \sqrt{3x+1}$ is said, “$H$ of $x$ is equal to the square root of $3x+1$.” We need to write this as $f$ of $g(x)$. Thus $f$ will be the square root function and $g(x)$ will simply be $3x + 1$. To make sure the function was decomposed correctly, remember that you can always checking be re-composing what you’ve decomposed. In the examples below, various decompositions are shown. Keep in mind that many decompositions exist for each example.

(a) $H(x) = \sqrt{3x+1}$

$$H(x) = \sqrt{3x+1}$$
$$f(x) = \sqrt{x}, \quad g(x) = 3x + 1$$

(b) $H(x) = (5x - 3)^2$

$$H(x) = (5x - 3)^2$$
$$f(x) = x^2, \quad g(x) = 5x - 3$$

(c) $H(x) = (x^2 - 1)^3$

$$H(x) = (x^2 - 1)^3$$
$$f(x) = x^3, \quad g(x) = x^2 - 1$$

(d) $H(x) = \frac{2}{x - 3}$

$$H(x) = \frac{2}{x - 3}$$
$$f(x) = \frac{2}{x}, \quad g(x) = x - 3$$

(e) $H(x) = \frac{\sqrt{x}}{\sqrt{x} + 1}$

$$H(x) = \frac{\sqrt{x}}{\sqrt{x} + 1}$$
$$f(x) = \frac{x}{x + 1}, \quad g(x) = \sqrt{x}$$
4.4 An Application to Function Composition

Example 9: A certain type of paint costs $30.50 per gallon and one gallon of paint will cover 250 sq. ft. Let $A$ be the number of square feet to be painted, let $n$ be the number of gallons of paint, and let $C$ be the cost of paint in dollars.

(a) Write the total cost of paint, $C$ (in dollars), as a function of the number of gallons of paint purchased, $n$. Call this function $f$.

Since the cost per gallon is $30.50, we write:

$$C = f(n) = 30.5n$$

(b) Write the number of gallons of paint needed, $n$, as a function of the number of square feet to be painted, $A$. Call this function $g$.

Since each 250 square feet require 1 gallon of paint, we will divide the total area by 250 to determine the necessary number of gallons:

$$A = g(n) = \frac{A}{250}$$

(c) Find and interpret $f(g(1000))$.

$$f(g(1000)) = f\left(\frac{1000}{250}\right) = f(4) = 30.5(4) = 122$$

The cost of painting 1000 square feet is $122.

(d) Find and interpret $f(g(A))$. 
\[ f(g(A)) = f\left(\frac{A}{250}\right) \]
\[ = 30.5 \left(\frac{A}{250}\right) \]
\[ = \frac{30.5}{250} A \]
\[ = \frac{61}{500} A \]

The expression \( f(g(A)) \) represents the cost of painting \( A \) square feet.

(e) Explain why \( g(f(n)) \) does not make sense.

The units don’t work out. The function \( g \) has area; the units for \( f(n) \) are dollars.