Example 1. Temperature in degrees Fahrenheit, $F$, can be written as a function of temperature in degrees Celsius, $C$. This relationship is given by $F = g(C) = \frac{9}{5}C + 32$.

(a) Find and interpret $g(100)$.

(b) Solve and interpret the solution to $g(C) = 32$.

(c) Solve the equation $F = \frac{9}{5}C + 32$ for $C$.

A function $f$ is said to be one-to-one if for every $y$-value in the range of $f$ there is exactly one $x$-value in the domain of $f$.

A function must be one-to-one in order to have an inverse. The inverse function of $f$ reverses the process of the original function. In other words, the input and output switch roles. The original function is given by $y = f(x)$. The inverse function is given by $x = f^{-1}(y)$. If we want to graph both of these functions in the $(x, y)$-plane, then we use $y = f^{-1}(x)$. To find the inverse, we switch the variables $x$ and $y$ and solve for $y$.

The inverse function of $f$ is denoted by $f^{-1}$. It is important to note that this notation is not denoting a reciprocal. That is, $f^{-1}(x) \neq \frac{1}{f(x)}$. 
Example 2. The function $f$ defined by $f(x) = 3x + 2$ is one-to-one. Find its inverse. Then graph $y = f(x)$ and $y = f^{-1}(x)$ in Figure 1. Include the graph of $y = x$ also. To verify that two functions are inverses, show that $f(f^{-1}(x)) = x$ and that $f^{-1}(f(x)) = x$.

The **horizontal line test** is a way of determining if a function is one-to-one. It states that if every horizontal line passes through a graph of a function at most once, then the function is one-to-one. In the same way that the vertical line test verifies if a graph represents a function, the horizontal line test verifies if the graph of a function is one-to-one (and thus invertible).

Example 3. Graph $f(x) = x^2 + 2$ in Figure 2. Then graph $g(x) = x^2 + 2, x \geq 0$ in Figure 3. Is either function invertible? Why or why not? If so, graph the inverse function.
Example 4. The function \( f \) defined by \( f(x) = -\frac{2x}{3x - 4} \) is one-to-one. Find the inverse function. Confirm that the inverse function you found is correct by showing \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

State the domain and range of each \( f \) and \( f^{-1} \).

The domain of \( f \) is the range of \( f^{-1} \). Similarly, the range of \( f \) is the domain of \( f^{-1} \).
Example 5. The function \( g \) defined by \( g(x) = x^3 - 8\sqrt[3]{x} + 8 \) is one-to-one. Find the inverse function and confirm that it is the inverse by showing \( g(g^{-1}(x)) = x \) and \( g^{-1}(g(x)) = x \). In Figure 4, use transformations to sketch \( y = g(x) \), \( y = g^{-1}(x) \) and \( y = x \).

![Figure 4](image-url)
Example 6. Use the functions $f$ and $g$ given in Table 1 to determine the following.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>9</td>
</tr>
</tbody>
</table>

(a) $g^{-1}(-2)$  (b) $f^{-1}(2)$  (c) $f^{-1}(0)$  (d) $f(g^{-1}(0))$

Example 7. Graph the inverse function of $f$ in Figure 5. Then use your sketch to find the values of $f^{-1}$ below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-2</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
</table>

(a) $f^{-1}(-4)$  (c) $f^{-1}(0)$  (e) $f^{-1}(4)$

(b) $f^{-1}(-2)$  (d) $f^{-1}(2)$
Example 8. The diameter of a Window-Pane oyster, $d$ (in mm), as a function of its weight, $w$ (in grams) can be modeled by

$$d = f(w) = 25 + 20w^{1/3}$$

Find the inverse function by solving $d = 25 + 20w^{1/3}$ for $w$. Write this inverse function as $g(d)$. 