Definition 1. A rational function is of the form $R(x) = \frac{p(x)}{q(x)}$ where $p$ and $q$ are polynomial functions.

The zeros of a rational function occur where $p(x) = 0$, as the function’s value is zero where the value of the numerator is zero.

A rational function is undefined where $q(x) = 0$, as the function is undefined whenever its denominator is zero.

The long run behavior of $R$ can be determined by the ratio of leading terms of $p$ and $q$.

Example 1. Determine the long-run behavior of the following functions.

(a) $R(x) = \frac{3}{x^2 - 4}$

$$\frac{3}{x^2} \quad \text{In the long run, } R \text{ "looks like" } \frac{3}{x^2}$$

(b) $R(x) = \frac{3x - 5}{x^2 + x - 6}$

$$\frac{3x}{x^2} = \frac{3}{x} \quad \text{In the long run, } R \text{ "looks like" } \frac{3}{x}$$

(c) $R(x) = \frac{x^2 - 5x - 6}{x^2 + x - 12}$

$$\frac{x^2}{x^2} = 1 \quad \text{In the long run, } R \text{ "looks like" } y = 1$$

(d) $R(x) = \frac{x^2 - 4x + 3}{x - 2}$

$$\frac{x^2}{x} = x \quad \text{In the long run, } R \text{ "looks like" } y = x$$

(e) $R(x) = \frac{3 - x^7}{4x^2 - 1}$

$$\frac{x^2}{4x^2} = -\frac{1}{4} x^5 \quad \text{In the long run, } R \text{ "looks like" } y = -\frac{1}{4} x^5$$

See Example 11
Example 2. Graph the rational function \( R(x) = \frac{x^2 - 4x + 3}{x - 3} \) by completing the following:

- Factor and simplify \( R(x) \). State the domain and any holes.
- State the long-run behavior and any horizontal asymptote. Determine if the function crosses its horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

\[
R(x) = \frac{x^2 - 4x + 3}{x - 3} = \frac{(x-1)(x-3)}{x-3} = x-1, \ x \neq 3
\]

Domain of \( R \): \((-\infty, 3) \cup (3, \infty)\)

\( \times \) There is a hole at 3
\( \times \) Place an open circle at \((3, 2)\)

\( \times \) Long Run Behavior:
\[
\frac{x^2}{x} = x
\]
"looks like" \( y = x \)
No horizontal asymptote

\( \times \) \( R(0) = -1 \)
\((0, -1)\)
\( \times \) \( \text{zeros: } 1 \), multiplicity 1
Example 3. Graph the rational function $R(x) = \frac{3x - 6}{x^2 + x - 6}$ by completing the following:

- Factor and simplify $R(x)$. State the domain and any holes.

- State the long-run behavior and any horizontal asymptote. Determine if the function crosses its horizontal asymptote.

- Find the vertical intercept.

- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

\[ R(x) = \frac{3x - 6}{x^2 + x - 6} \]

\[ = \frac{3(x-2)}{(x+3)(x-2)} \]

\[ = \frac{3}{x+3}, \quad x \neq 2 \]

**Domain of R:** \( \{ x \mid x \neq 2, -3 \} \)

**There is a hole at** \( x = 2 \)

**The y-value is** \( \frac{3}{3} = 1 \)

**Long Run Behavior:**

\[ \frac{3x}{x^2} = \frac{3}{x} \]

**Horizontal Asymptote:** \( y = 0 \)

**Does R cross its horizantl asymptote?**

\[ 0 = \frac{3x - 6}{x^2 + x - 6} \]

\[ 0(x^2 + x - 6) = 3x - 6 \]

\[ 0 = 3x - 6 \]

\[ 0 = 3x \]

\[ x = x \]

**It crosses at** \( x = 2 \).

**There is a hole at** \( 2 \).
Example 4. Graph the rational function \( R(x) = \frac{x^2 - 5x - 6}{x^2 + x - 12} \) by completing the following:

- Factor and simplify \( R(x) \). State the domain and any holes.
- State the long-run behavior and any horizontal asymptote. Determine if the function crosses its horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

\[
R(x) = \frac{x^2 - 5x - 6}{x^2 + x - 12} = \frac{(x-6)(x+1)}{(x-3)(x+4)}
\]

Domain: \( \{ x \mid x \neq 3, -4 \} \)

\[ \frac{x^2}{x^2} = 1 \]

Horizontal Asymptote: \( y = 1 \)

Does \( R \) cross \( y = 1 \)?

\[
1 = \frac{x^2 - 5x - 6}{x^2 + x - 12}
\]

\[
x^2 + x - 12 = x^2 - 5x - 6
\]

\[
x - 12 = -5x - 6
\]

\[
x = -6
\]

\[
1 = x
\]

Crosses at \( x = 1 \) (y-value is 2)

\[ \boxed{\text{Vertical Intercept}} R(0) = \frac{0^2 - 5(0) - 6}{0^2 + 0 - 12} = \frac{1}{2} (0, \frac{1}{2}) \]

\[ \boxed{\text{Zeros: } 6, -1} \]

\[ \boxed{\text{Vertical Asymptotes: } x = 3, x = -4} \]

\[ \boxed{\text{Numerator!}} \]

\[ \boxed{\text{Denominator!}} \]

\[
\begin{array}{c|cccccc}
\text{Interval} & (-\infty, -4) & (-4, -1) & (-1, 3) & (3, 6) & (6, \infty) \\
\hline
x & -6 & -2 & 0 & 4 \\
R(x) & R(-6) = \frac{10}{5} & R(-2) = \frac{2}{5} & R(0) = 1 & R(4) = \frac{-5}{4} \\
+/- & + & (above) & - & (below) & + & - \\
\text{Pit point} & (-6, \frac{10}{5}) & (-2, -\frac{4}{5}) & (0, 1) & (4, \frac{-5}{4})
\end{array}
\]

\[ \text{Figure 3} \]

Instructor: A.E. Cary
Example 5. Graph the rational function \( R(x) = \frac{8}{x^2 - 4} \) by completing the following:

- Factor and simplify \( R(x) \). State the domain and any holes.
- State the long-run behavior and any horizontal asymptote. Determine if the function crosses its horizontal asymptote.
- Find the vertical intercept.
- Find any zeros and find any vertical asymptotes. State the behavior of the function around the zeros and vertical asymptotes (preferably by making a table).

\[
R(x) = \frac{8}{x^2 - 4} = \frac{8}{(x-2)(x+2)}
\]

Domain of \( R(x) \): \( \{ x \mid x \neq 2, -2 \} \)

No holes!

**Long-Run Behavior:**

\[
\frac{8}{x^2}
\]

Horizontal Asymptote: \( y = 0 \)

Does it cross?

\[
0 = \frac{8}{x^2 - 4}
\]

\[
0(x^2 - 4) = 8
\]

\[
0 = 8 \quad \text{Lies!}
\]

No solution.

It does not cross.

\[
R(0) = \frac{8}{0^2 - 4} = -2
\]

The vertical intercept is \( (0, -2) \)

Zeros: None.

Vertical Asymptotes: \( x = 2, x = -2 \)

(Denominator)

*Note: We drew the behavior b/n -2 and 2 based on the fact that the function does not cross \( y = 0 \). You could also evaluate \( R(x) \) at -1 and 1.*

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Interval} & (-\infty, -2) & \text{Skipped} & (2, \infty) \\
\hline
x & -4 & & 4 \\
\hline
R(x) & 0(4) = \frac{2}{3} & & R(4) = \frac{2}{3} \\
\hline
+/- & + & & + \\
\hline
\text{Plot points} & (-4, \frac{2}{3}) & & (4, \frac{2}{3}) \\
\end{array}
\]

**Figure 4**

\[
\text{Graph of } \frac{8}{x^2 - 4}
\]
$$f(x) = 3(x-7)(x+3)^2$$

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
<th>Behavior near zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>straight through</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>bounce / touch</td>
</tr>
</tbody>
</table>

- Degree: 3  
- Max number of turning points: 2

- In the long run, $f(x)$ looks like $3x^3$.

- As $x \to \infty$, $f(x) \to \infty$
- As $x \to -\infty$, $f(x) \to -\infty$

**Alternate notation:**
- $\lim_{x \to \infty} f(x) = \infty$
- $\lim_{x \to -\infty} f(x) = -\infty$

- **y-intercept:**
  - $f(0) = 3(0-7)(0+3)^2$
  - $= -189$