In this section, we will explore function transformations. We will explore these numerically (in tabular form), algebraically (as formulas), and graphically. When you studied the vertex form of a parabola, you were actually studying function transformations for a specific function—namely, \( f(x) = x^2 \). For example, when graphing \( y = -(x - 6)^2 - 3 \), you know that the graph points downward and that the vertex is \((6, -3)\).

We could also say that the graph is reflected about the \(x\)-axis, shifted right 6 units, and then shifted down 3 units. In this course, we will be able to apply similar transformations to any function—not just parabolas! One such example is shown below.

http://www.esrl.noaa.gov/gmd/ccgg/trends/
Let \( y = f(x) \), where \( x \) is the number of months after January 1, 2011 and \( f(x) \) is the amount of CO\(_2\) in the atmosphere after \( x \) months. We will measure \( f(x) \) in parts per million above 380 and restrict \( x \) to \(-3 \leq x \leq 9\). The data for September 2010 through September 2011 is shown in Figure 2.

**Vertical Shifts**

**Example 1.** Complete Table 1 using the function values for \( f \). What happens to the graph in each case? Sketch and label the graph of \( y = f(x) + 4 \) and the graph of \( y = f(x) - 2 \) in Figure 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>( f(x) + 4 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) - 2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Summary of Vertical Shifts**

The graph of \( y = f(x) + k \) is transformation of the graph of \( y = f(x) \).
- If \( k > 0 \), then the graph of the original function shifts _______ by \( k \) units.
- If \( k < 0 \), then the graph of the original function shifts _______ by \( k \) units.


**Horizontal Shifts**

Horizontal shifts are not quite as straightforward as vertical shifts. The primary reason is that in order to shift the graph horizontally, we need to add or subtract from \( x \)—before we evaluate the function. The end result is that horizontal transformations work a bit backwards from what you may expect, as we will discover in the example below.

**Example 2.** Complete Table 2 using the function values for \( f \). What happens to the graph in each case? Sketch and label the graph of \( y = f(x + 3) \) and the graph of \( y = f(x - 6) \) in Figure 3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-6</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>und.</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>11</td>
<td>9</td>
<td>und.</td>
<td>und.</td>
</tr>
<tr>
<td>( f(x + 3) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x - 6) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Summary of Horizontal Shifts**

The graph of \( y = f(x + h) \) is transformation of the graph of \( y = f(x) \).

- If \( h > 0 \), then the graph of the original function shifts ________ by \( h \) units.
- If \( h < 0 \), then the graph of the original function shifts ________ by \( h \) units.
Example 3. For each function below, the “original” or “basic” function is $y = |x|$. Use the properties of horizontal and vertical shifts to graph the stated transformations. The full graph and 3 key points are given in each.

(a) Graph $y = |x| - 5$.

(b) Graph $y = |x + 4|$.

(c) Graph $y = |x + 2| - 1$.

(d) Graph $y = |x - 3| - 6$. 
**Vertical Stretches and Compressions**

**Example 4.** Assume the base temperature setting for the thermostat in a house is 64°F. Let \( g(x) \) be the number of degrees above 64°F \( x \) hours after 6am. Complete Table 3 using the function values for \( g \). What happens to the graph in each case? Sketch and label the graph of \( y = 2g(x) \) in Figure 8 and the graph of \( y = \frac{1}{2}g(x) \) in Figure 9.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>4</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>-2</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>( 2g(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2}g(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Summary of Vertical Stretches and Compressions**
The graph of \( y = Af(x) \) is transformation of the graph of \( y = f(x) \). If

- If \(|A| > 1\), then the graph of the original function ________________ vertically by a factor of \(|A|\).
- If \(0 < |A| < 1\), then the graph of the original function ________________ vertically by a factor of \(|A|\).
**Horizontal Stretches and Compressions**

Horizontal stretches and compressions, much like horizontal shifts, work in a somewhat counterintuitive way. This again is a result of the fact that we will multiply $x$ by a number before we evaluate the function.

**Example 5.** The graph of $y = h(x)$ is shown below. Complete Table 4 and then graph $y = h\left(\frac{1}{2}x\right)$ in Figure 10.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-12</th>
<th>-8</th>
<th>-6</th>
<th>-4</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
<td>und.</td>
<td>und.</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>-4</td>
<td>6</td>
<td>und.</td>
</tr>
<tr>
<td>$h\left(\frac{1}{2}x\right)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 10**

![Graph of $y = h(x)$ and $y = h\left(\frac{1}{2}x\right)$]
Example 6. The graph of $y = h(x)$ is shown below. Complete Table 5 and then graph $y = h(4x)$ in Figure 11. An “X” is placed where the function is defined but difficult to evaluate.

**Table 5**

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-4</th>
<th>-1.5</th>
<th>-1</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
<td>0</td>
<td>0</td>
<td>X</td>
<td>3</td>
<td>4</td>
<td>X</td>
<td>2</td>
<td>0</td>
<td>-4</td>
<td>6</td>
</tr>
<tr>
<td>$h(4x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Summary of Horizontal Stretches and Compressions**

The graph of $y = f(Bx)$ is transformation of the graph of $y = f(x)$.

- If $|B| > 1$, then the graph of the original function _________ horizontally by a factor of $\frac{1}{|B|}$.
- If $0 < |B| < 1$, then the graph of the original function _________ horizontally by a factor of $\frac{1}{|B|}$.
**Horizontal and Vertical Reflections**

**Example 7.** The graph of \( y = h(x) \) is shown below. Complete Table 6 and then graph \( y = -h(x) \) in Figure 12 and graph \( y = h(-x) \) in Figure 13.

**Table 6**

<table>
<thead>
<tr>
<th>( x )</th>
<th>-8</th>
<th>-6</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) )</td>
<td>und.</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>-4</td>
<td>6</td>
</tr>
<tr>
<td>(-h(x))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h(-x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 12**

**Figure 13**

**Summary of Horizontal and Vertical Reflections**

- The graph of \( y = -f(x) \) is transformation of the graph of \( y = f(x) \). It reflects the graph of the original function across the ___________ axis.

- The graph of \( y = f(-x) \) is transformation of the graph of \( y = f(x) \). It reflects the graph of the original function across the ___________ axis.
Example 8. For each function below, the “original” or “basic” function is $y = \sqrt{x}$. Use the properties of horizontal and vertical stretches and compressions to graph the stated transformations. The full graph and 4 key points are given in each.

(a) Graph $y = 4\sqrt{x}$.

(b) Graph $y = \sqrt{\frac{1}{3}x}$.

(c) Graph $y = -\sqrt{x}$.

(d) Graph $y = \frac{1}{2}\sqrt{x}$.

(e) Graph $y = \sqrt{2x}$.

(f) Graph $y = \sqrt{-x}$. 

Instructor: A.E.Cary
Example 9. The point $(4, 12)$ is on the graph of $y = f(x)$. Determine the point on the graph of...

(a) $y = f(x + 2) - 1$

(b) $y = 5f(x)$

(c) $y = -f(x - 5) + 4$

(d) $y = f \left( \frac{1}{3}x \right)$

(e) $y = f(-x) - 5$

(f) $y = 2f(4(x + 1)) - 3$
Example 10. For the function below, identify the original (or “basic”) function and explain how the graph is a transformation of the graph of the original function. State all steps to this transformation in an appropriate order.

(a) $g(x) = 8\sqrt[3]{-4x}$

(b) $h(x) = -|2x + 6|

(c) $j(x) = \frac{2}{3} (5(x - 1))^3 + 4$
Example 11. Let \( g(x) = -(x - 6)^2 - 3 \).

(a) Identify the original (or “basic”) function and explain how the graph of \( y = g(x) \) is a transformation of the original function. State all steps to this transformation in an appropriate order.

(b) Compare the graph of \( y = g(x) \) to the graph of \( y = x^2 \) after it has been shifted right 6 units, shifted down 3 units and THEN reflected about the \( x \)-axis.
Example 12. Let \( g(x) = \frac{1}{2}(x + 5)^3 + 4 \). Identify the original function and explain how the graph of \( y = g(x) \) is a transformation of the graph of the original function. Then sketch a graph of \( y = g(x) \) in Figure 22.

Example 13. Let \( g(x) = \left| \frac{1}{2}x - 3 \right| - 1 \). Identify the original function and explain how the graph of \( y = g(x) \) is a transformation of the graph of the original function. Then sketch a graph of \( y = g(x) \) in Figure 23.
Example 14. Let \( g(x) = \sqrt{-(x + 3)} + 2 \). Identify the original function and explain how the graph of \( y = g(x) \) is a transformation of the graph of the original function. Then sketch a graph of \( y = g(x) \) in Figure 24.

![Figure 24](image1)

Example 15. Let \( g(x) = -f(2(x + 4)) + 3 \). The original function \( y = f(x) \) is shown in Figure 25. Explain how the graph of \( y = g(x) \) is a transformation of the graph of the original function. Then sketch a graph of \( y = g(x) \) in Figure 25.

![Figure 25](image2)
**Group Work.** Complete the following for each set of functions below that your group is assigned:

- Identify and graph the basic function used in this transformation. (Example: \( f(x) = x^2 \)). Use your Library of Functions Handout if necessary.
- State the series of transformations and the order in which they occur.
- Graph the transformation.
- Check your work. This can be done by hand by creating a table or with your graphing calculator.

### Transformations

**Section I: Horizontal and Vertical Shifts**

(a) \( g_1(x) = (x - 5)^2 + 1 \)

(b) \( g_2(x) = \sqrt{x + 4} + 2 \)

(c) \( g_3(x) = (x + 1)^3 - 2 \)

(d) \( g_4(x) = \frac{1}{x - 2} + 3 \)

(e) \( g_5(x) = |x + 8| - 6 \)

(f) \( g_6(x) = \sqrt{x - 4} - 2 \)

**Section II: Horizontal and Vertical Stretches and Reflections**

(a) \( g_1(x) = \sqrt{-2x} \)

(b) \( g_2(x) = -5\sqrt{x} \)

(c) \( g_3(x) = \left(-\frac{1}{2}x\right)^3 \)

(d) \( g_4(x) = -\frac{1}{2}x^3 \)

(e) \( g_5(x) = -3x^2 \)

(f) \( g_6(x) = |5x| \)

**Section III: Combined Function Transformations**

(a) \( g_1(x) = 2|x| - 3 \)

(b) \( g_2(x) = -(x + 1)^3 - 3 \)

(c) \( g_3(x) = \sqrt{-x} + 4 \)

(d) \( g_4(x) = 3(x - 2)^2 + 5 \)

(e) \( g_5(x) = \frac{2}{x} + 5 \)

(f) \( g_6(x) = 4\sqrt{2(x + 1)} + 3 \)