Supplemental Solutions for the Related Rates Lab

Exercise 8.1

Define \( x \) to be the elevation (ft) of the balloon \( t \) seconds after the balloon begins to rise and \( y \) to be the distance (ft) between the observer and the balloon at the same instant.

The relation equation is \( x^2 + 300^2 = y^2 \).

The rate equation is:

\[
\frac{d}{dt}(x^2 + 90000) = \frac{d}{dt}(y^2) \Rightarrow 2x \frac{dx}{dt} = 2y \frac{dy}{dt}
\]

When the elevation of the balloon is 400 feet:

\[ x = 400 \text{, } y = 500 \text{ (from the Pythagorean Theorem), and } \frac{dx}{dt} = 10 \]

Substituting these values into the rate equation we get:

\[ 2(400)(10) = 2(500) \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = 8 \]

So the distance between the observer and the balloon is increasing at a rate of 8 ft/s at the instant the elevation of the balloon is 400 feet.
Exercise 8.2

Define \( x \) to be the vertical distance (ft) between the tip of the arm and the horizontal position of the arm \( t \) seconds after gate begins to close and \( \theta \) to be the angle of elevation (rad) at the pivot point at the same instant in time.

The relation equation is \( \sin(\theta) = \frac{x}{28} \).

The rate equation is:

\[
\frac{d}{dt}(\sin(\theta)) = \frac{d}{dt}\left(\frac{x}{28}\right) \Rightarrow \cos(\theta) \frac{d\theta}{dt} = \frac{1}{28} \frac{dx}{dt}
\]

At the instant the angle at the pivot point is \(30^\circ\):

\[
\theta = \frac{\pi}{6} \quad \text{and} \quad \frac{d\theta}{dt} = \left(-\frac{6 \text{ deg}}{s}\right)\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = -\frac{\pi}{30} \text{ rad/s}
\]

Substituting these values into the rate equation we have:

\[
\cos\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{30}\right) = \frac{1}{28} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} \approx -2.54
\]

So at the instant the angle of elevation of the arm is \(30^\circ\), the tip of the arm is approaching the ground at a rate of about 2.54 ft/s.
Exercise 8.3

Define $V'$ to be the volume of soda (cm$^3$) that remains in Jimbo’s cup $t$ seconds after he commences to sip and $h$ to be the height of the soda in the cup (cm) at that very same instant.

The volume formula for a right circular cone is

$$V = \frac{\pi}{3} r^2 h$$

where $V$ represents the volume of the cone, $h$ represents the height of the cone, and $r$ represents the radius at the top of the cone. Since neither of our defined variable is a radius, we need to purge that variable from our volume formula.

From the similar triangles shown in Figure E8.3K we have,

$$\frac{5}{10} = \frac{r}{h} \Rightarrow r = \frac{h}{2}.$$

Substituting the expression $\frac{h}{2}$ for $r$ in the volume formula we get our relation equation:

$$V = \frac{\pi}{3} \frac{h^3}{4} \Rightarrow V = \frac{\pi}{12} h^3$$

Our rate equation is:

$$\frac{d}{dt}(V) = \frac{d}{dt} \left( \frac{\pi}{12} h^3 \right) \Rightarrow \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

At the instant there are 100 cm$^3$ of soda remaining in the cup:

$$100 = \frac{\pi}{12} h^3 \Rightarrow h = \sqrt[3]{\frac{1200}{\pi}}; \text{ also, } \frac{dV}{dt} = -0.25$$

Substituting these values into our rate equation we get:

$$-0.25 = \frac{\pi}{4} \left( \frac{1200}{\pi} \right)^{2/3} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} \approx -0.006$$

So at the instant there are 100 cm$^3$ of soda remaining in the cup, the height of the soda in the cup is decreasing at a rate of about 0.006 cm/s. Somebody needs to tell Jimbo to put some juice into his sipping rate!
Exercise 8.4

Define $V$ to be the volume of the snowball (cm³) and $A$ to be the surface area of the snowball (cm²) $t$ minutes after the snowball began to melt.

The volume and surface area formulas for a sphere in terms of the radius, $r$, of the sphere are, respectively, $V = \frac{4}{3} \pi r^3$ and $A = 4 \pi r^2$. Solving the volume formula for $r$ we have $r = \left(\frac{3}{4 \pi}\right)^{1/3} V^{1/3}$ and substituting the resultant expression into the area formula we have our relation equation:

$$A = 4 \pi r^2 \Rightarrow A = 4 \pi \left(\frac{3^{1/3}}{(4 \pi)^{1/3}} V^{1/3}\right)^2 \Rightarrow A = \sqrt[3]{36 \pi} V^{2/3}$$

This gives us our rate equation:

$$\frac{dA}{dt} = \frac{d}{dt}\left(\sqrt[3]{36 \pi} V^{2/3}\right) \Rightarrow \frac{dA}{dt} = 2 \sqrt[3]{36 \pi} V^{-1/3} \frac{dV}{dt} \Rightarrow \frac{dA}{dt} = \frac{2}{3} \sqrt[3]{36 \pi} \frac{dV}{dt}$$

When the radius of the snowball is 6 cm:

$$V = \frac{4}{3} \pi (6)^3 \quad \text{and} \quad \frac{dV}{dt} = -25$$

$$= 288 \pi$$

Substituting these values into our rate equation we get:

$$\frac{dA}{dt} = \frac{2}{3} \sqrt[3]{36 \pi} \left(-25\right)$$

$$= -8 \sqrt[3]{\pi}$$

So at the instant the radius of the snowball is 6 cm, the surface area of the snowball is decreasing at the rate of $8 \sqrt[3]{\pi}$ cm²/minute.