Key Concepts: Introduction to rational functions
Simplifying rational expressions
Multiplying and dividing rational expressions

Definition

A function f whose formula can be written in the form $f(x)=\dfrac{p(x)}{q(x)}$ where p and q are both polynomial functions is called <u>a rational function.</u>

Example 1

Graph the function $y=\dfrac{x-3}{x^2+x-12}$ on your calculator and carefully copy the graph onto Figure 1. State the domain of the function. Finally, write the equation for the function in simplified form.

$$y=\frac{x-3}{x^2+x-12}$$

$$y=\frac{x-3}{(x+4)(x-3)}$$

$$y=\frac{1}{x+4}\cdot\frac{x-3}{x-3}$$

$$y=\frac{1}{x+4}\;;\quad x\neq 3$$

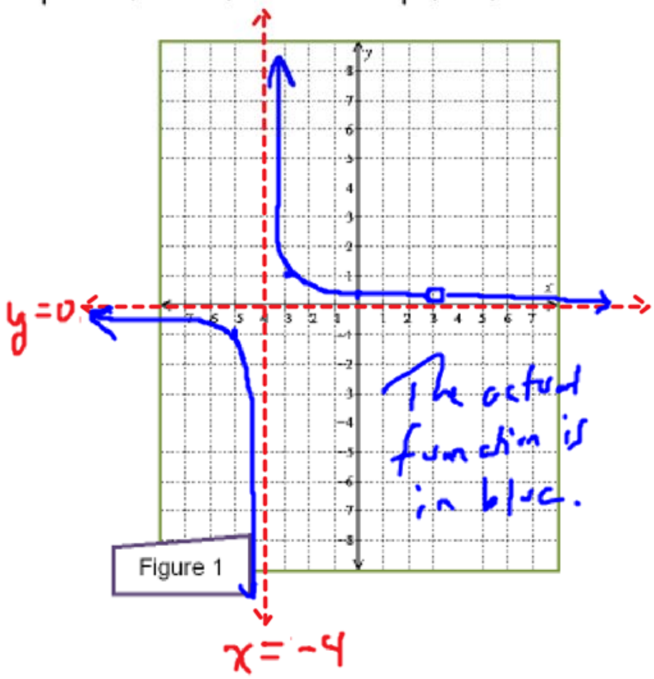

Figure 1

The domain is $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$

$x=-4$ is a vertical asymptote.

$y=0$ is a horizontal asymptote.

Asymptotes are not part of the function, they are visual guides.

Example 2

Graph the function $y = \dfrac{10}{t^2 + 2t - 8}$ on your calculator and carefully copy the graph onto Figure 2. State the domain of the function. Finally, write the equation for the function in simplified form.

$$y = \frac{10}{t^2+2t-8}$$

$$y = \frac{10}{(t+4)(t-2)}$$

$t+4=0$
$t=-4$

$t-2=0$
$t=2$

The domain is $(-\infty,-4)\cup(-4,2)\cup(2,\infty)$

Vertical asymptotes: $t=-4$, $t=2$

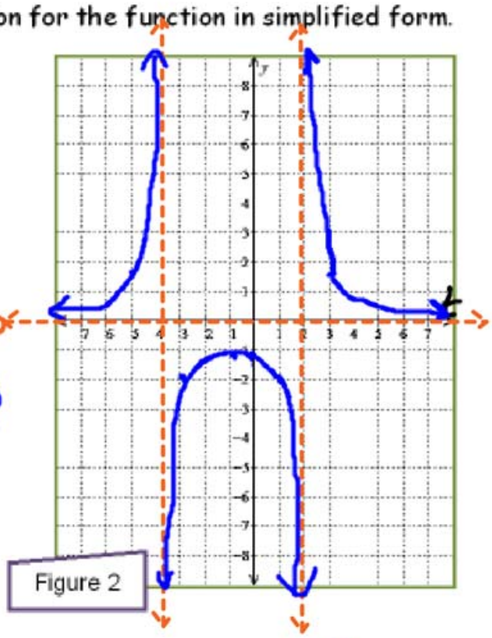

Figure 2

Example 3

Simplify the formula for $g(x) = \dfrac{x(x-5)(2x+1)}{x(2x-1)(x-5)(x+7)}$. Make sure that you state any necessary domain restrictions. State any other numbers that are not in the domain of g.

$$g(x) = \frac{x(x-5)(2x+1)}{x(2x-1)(x-5)(x+7)}$$

$$= \frac{2x+1}{(2x-1)(x+7)} \cdot \frac{x}{x} \cdot \frac{x-5}{x-5}$$

$$= \frac{2x+1}{(2x-1)(x+7)}\ ;\ x \neq 0,\ x \neq 5$$

$2x-1=0$
$x+7=0$
$x=0$
$x-5=0$

$2x-1=0$
$x+7=0$

The graph would have holes at 0 and 5.

The vertical asymptotes would be $x = \frac{1}{2}$ and $x=-7$.

Example 4

Simplify $\dfrac{x^2-5x+4}{2x^2+5x-12}$. Make sure that you state any necessary domain restrictions.

$$\frac{x^2-5x+4}{2x^2-5x-12} = \frac{(x-4)(x-1)}{(2x+3)(x-4)}$$

Graph has a hole at 4.

$$= \frac{x-1}{2x+3} \cdot \frac{x-4}{x-4}$$

$x = -\dfrac{3}{2}$ is the vertical asymptote

$$= \frac{x-1}{2x+3} \; ; \; x \neq 4$$

Recall

To factor $2x^2+5x-12$ we first need to find an integer factor pair of $(2)(-12)$ that adds to 5

Factor pairs of -24

$1, -24$

$2, -12$

$3, -8$ ← close need $-3, 8$

$$\begin{aligned} 2x^2+5x-12 &= 2x^2-3x+8x-12 \\ &= x(2x-3)+4(2x-3) \\ &= (2x-3)(x+4) \end{aligned}$$

Example 4

Simplify each rational expression; make sure that you state any necessary restrictions to the domains.

Simplify $\dfrac{5t+30}{40-10t}$.

$$\frac{5t+30}{40-10t} = \frac{5(t+6)}{10(4-t)}$$
$$= \frac{5}{5} \cdot \frac{t+6}{2(4-t)}$$
$$= \frac{t+6}{2(4-t)}$$

Simplify $\dfrac{x^2+7x}{x^2-7x}$.

$$\frac{x^2+7x}{x^2-7x} = \frac{x(x+7)}{x(x-7)}$$
$$= \frac{x}{x} \cdot \frac{x+7}{x-7}$$
$$= \frac{x+7}{x-7}; \ x \neq 0$$

NOTE

On a graph of $y = \dfrac{x^2+7x}{x^2-7x}$, $x = 7$ is a vertical asymptote & there's a hole at $x=0$.

Simplify $\dfrac{x^4+21x^2}{5x^3+x^7}$.

$$\frac{x^4+21x^2}{5x^3+x^7} = \frac{x^2(x^2+21)}{x^3(5+x^4)}$$
$$= \frac{x^2}{x^2} \cdot \frac{x^2+21}{x(5+x^4)}$$
$$= \frac{x^2+21}{x(5+x^4)}$$

Example 5

Simplify $\frac{t^3-8}{t^2-4t+4}$. State any necessary restrictions on the domain.

$$\frac{t^3-8}{t^2-4t+4} = \frac{(t-2)(t^2+2t+4)}{(t-2)(t-2)}$$

$$= \frac{t-2}{t-2} \cdot \frac{t^2+2t+4}{t-2}$$

$$= \frac{t^2+2t+4}{t-2}$$

$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$t^3-8=(t-2)(t^2+2t+4)$$

$a=t$ $\quad t^3-8=t^3-2^3$

$b=2$

Example 6

Simplify the formula for $f(y)=\frac{y^2-10y-56}{y^2+16}$. What is the domain of f?

$$f(y) = \frac{y^2-10y-56}{y^2+16}$$

$$= \frac{(y-14)(y+4)}{y^2+16}$$

The domain of f is $(-\infty, \infty)$.
cuz no number created division by 0.

$a^2 - b^2 = (a+b)(a-b)$

Example 7

Simplify the formula for $g(t) = \frac{4t^2 - 9}{2t-3}$. State any necessary restrictions on the domain. What other numbers are not in the domain of g?

$2t - 3 \neq 0$
$2t \neq 3$
$t \neq 3/2$

$$g(t) = \frac{4t^2 - 9}{2t-3}$$
$$= \frac{(2t+3)(2t-3)}{2t-3}$$
$$= \frac{2t-3}{2t-3} \cdot \frac{2t+3}{1}$$
$$= 2t+3 \; ; \; t \neq 3/2$$

domain restriction.

3/2 is the only number not in the domain of g

Example 8

Simplify $\frac{10n+15}{n^2-1} \cdot \frac{n+1}{6n+9}$.

$$\frac{10n+15}{n^2-1} \cdot \frac{n+1}{6n+9} = \frac{(10n+15)(n+1)}{(n^2-1)(6n+9)}$$
$$= \frac{5(2n+3)(n+1)}{(n+1)(n-1)\cdot 3(2n+3)}$$
$$= \frac{2n+3}{2n+3} \cdot \frac{n+1}{n+1} \cdot \frac{5}{3(n-1)}$$
$$= \frac{5}{3(n-1)} \; ; \; n \neq -\frac{3}{2} \; ; \; n \neq -1$$

Side note

on a graph of
$y = \frac{10x+15}{x^2-1} \cdot \frac{x+1}{6x+9}$
The vertical asymptote would be $x = 1$
and there'd be holes
at $x = -\frac{3}{2}$ and $x = -1$.

Sum of cubes

$a = x \quad a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$b = 3 \quad x^3 + 27 = (x+3)(x^2 - 3x + 9)$

Example 9

Simplify $\left(x^2-4\right)\cdot\dfrac{x+2}{4x-8}$.

$$(x^2-4)\cdot\frac{x+2}{4x-8} = \frac{x^2-4}{1}\cdot\frac{x+2}{4x-8}$$

$$= \frac{(x^2-4)(x+2)}{4x-8}$$

$$= \frac{(x+2)(x-2)(x+2)}{4(x-2)}$$

$$= \frac{x-2}{x-2}\cdot\frac{(x+2)(x+2)}{4}$$

$$= \frac{(x+2)^2}{4};\ x \neq 2$$

Example 10

Simplify $\dfrac{4x^4+20x^3+24x^2}{x^2-25}\cdot\dfrac{5x+25}{x^3+27}$.

$$\frac{4x^4+20x^3+24x^2}{x^2-25}\cdot\frac{5x+25}{x^3+27} = \frac{(4x^4+20x^3+24x^2)(5x+25)}{(x^2-25)(x^3+27)}$$

$$= \frac{4x^2(x^2+5x+6)\cdot 5(x+5)}{(x+5)(x-5)(x+3)(x^2-3x+9)}$$

$$= \frac{4x^2(x+2)\cancel{(x+3)}\cdot 5\cancel{(x+5)}}{\cancel{(x+5)}(x-5)\cancel{(x+3)}(x^2-3x+9)}$$

$$= \frac{x+5}{x+5}\cdot\frac{x+3}{x+3}\cdot\frac{(4)(5)(x^2)(x+2)}{(x-5)(x^2-3x+9)}$$

$$= \frac{20x^2(x+2)}{(x-5)(x^2-3x+9)};\ x\neq -5;\ x\neq -3$$

Example 11

Simplify $\dfrac{x^2-6x-7}{2x+2} \div (x-7)$.

$$\frac{x^2-6x-7}{2x+2} \div (x-7) = \frac{\frac{x^2-6x-7}{2x+2}}{\frac{x-7}{1}}$$

$$= \frac{x^2-6x-7}{2x+2} \cdot \frac{1}{x-7}$$

$$= \frac{(x-7)(x+1)}{2(x+1)(x-7)}$$

$$= \frac{x-7}{x-7} \cdot \frac{x+1}{x+1} \cdot \frac{1}{2}$$

$$= \frac{1}{2};\ x \neq 7;\ x \neq -1$$

Example 12

Simplify $\dfrac{2x}{x-2} \div \dfrac{x+2}{x} \div \dfrac{7x}{x^2-4}$.

$$\frac{2x}{x-2} \div \frac{x+2}{x} \div \frac{7x}{x^2-4} = \frac{2x}{x-2} \cdot \frac{x}{x+2} \div \frac{7x}{x^2-4}$$

$$= \frac{2x}{x-2} \cdot \frac{x}{x+2} \cdot \frac{x^2-4}{7x}$$

$$= \frac{2x^2(x+2)(x-2)}{7x(x-2)(x+2)}$$

$$= \frac{x}{x} \cdot \frac{x+2}{x+2} \cdot \frac{x-2}{x-2} \cdot \frac{2x}{7}$$

$$= \frac{2x}{7};\ x \neq -2;\ x \neq 2;\ x \neq 0$$