Key Concepts: Rational exponents
Working with radicals!

Expressing roots and radicals with exponents

Assuming that a is in the domain and n is a positive integer, $\sqrt[n]{a} = a^{1/n}$

Example 1

Express each radical expression using a rational exponent.

$\sqrt[3]{4}$

$\sqrt[3]{4} = 4^{1/3}$

$\sqrt[7]{-8}$

$\sqrt[7]{-8} = (-8)^{1/7}$

$\sqrt{6x}$

$\sqrt{6x} = (6x)^{1/2}$

$\sqrt{x^3}$

$\sqrt{x^3} = (x^3)^{1/2}$
$= x^{3 \cdot \frac{1}{2}}$
$= x^{3/2}$

$\sqrt[5]{(x-8)^6}$

$\sqrt[5]{(x-8)^6} = \left[(x-8)^6\right]^{1/5}$
$= (x-8)^{6 \cdot \frac{1}{5}}$
$= (x-8)^{6/5}$

$\sqrt[3]{x^{12}}$

$\sqrt[3]{x^{12}} = x^{12/3}$
$= x^4$

Recall: $(a^m)^n = a^{mn}$

Example 2

Express each exponential expression in radical form.

Assuming that a is in the domain and m and n are positive integers, $a^{m/n} = \sqrt[n]{a^m}$

$x^{3/2}$

$$x^{3/2} = \sqrt[2]{x^3} = \sqrt{x^3}$$

$t^{2/3}$

$$t^{2/3} = \sqrt[3]{t^2}$$

$(5y)^{7/11}$

$$(5y)^{7/11} = \sqrt[11]{(5y)^7}$$

$8w^{4/9}$

$$8w^{4/9} = 8\sqrt[9]{w^4}$$

$v^{-1/2}$

$$v^{-1/2} = \frac{1}{v^{1/2}} = \frac{1}{\sqrt{v}}$$

$12k^{-7/6}$

$$12k^{-7/6} = 12 \cdot \frac{1}{k^{7/6}} = \frac{12}{\sqrt[6]{k^7}}$$

$-q^{-1/3}$

$$-q^{-1/3} = -\frac{1}{q^{1/3}} = -\frac{1}{\sqrt[3]{q}}$$

$(4x)^{-8/19}$

$$(4x)^{-8/19} = \frac{1}{(4x)^{8/19}} = \frac{1}{\sqrt[19]{(4x)^8}}$$

Recall: $a^{-p} = \dfrac{1}{a^p}$

does $\sqrt[5]{t^{20}} = t^4$ make sense?

$(t^4)^5 = t^{(4)5}$
$= t^{20}$ yep!

$a^{m/n} \to (a^m)^{1/n} = \sqrt[n]{a^m}$
$a^{m/n} \to (a^{1/n})^m = (\sqrt[n]{a})^m$

Example 3

Simplify each radical expression after first rewriting the expression in exponential form.

$\sqrt[5]{t^{20}}$

$\sqrt[5]{t^{20}} = t^{20/5}$
$= t^4$

$6\sqrt[33]{x^{77}}$

$6\sqrt[33]{x^{77}} = 6x^{77/33}$
$= 6x^{7/3}$
$= 6\sqrt[3]{x^7}$

$\left(\sqrt{3}\right)^{10}$

$\left(\sqrt{3}\right)^{10} = 3^{10/2}$
$= 3^5$
$= 243$

$\sqrt[4]{9^2}$

$\sqrt[4]{9^2} = 9^{2/4}$
$= 9^{1/2}$
$= \sqrt{9}$
$= 3$

$\sqrt{w}\,\sqrt[4]{w}$

$\sqrt{w}\,\sqrt[4]{w} = w^{1/2}w^{1/4}$
$= w^{\frac{1}{2}+\frac{1}{4}}$
$= w^{3/4}$
$= \sqrt[4]{w^3}$

$\sqrt[7]{x^6}\,\sqrt[7]{x}$

$\sqrt[7]{x^6}\,\sqrt[7]{x} = x^{6/7}x^{1/7}$
$x^{6/7+1/7}$
$= x$
$= x^1$
$= x$

$\left(\sqrt[12]{x^7 y^{16}}\right)^{36}$

$\left(\sqrt[12]{x^7y^{16}}\right)^{36} = (x^7y^{16})^{36/12}$
$= (x^7y^{16})^3$
$= (x^7)^3(y^{16})^3$
$= x^{21}y^{48}$

$\sqrt[5]{\sqrt[3]{x^{15}}}$

$\sqrt[5]{\sqrt[3]{x^{15}}} = \sqrt[5]{x^{15/3}}$
$= \sqrt[5]{x^5}$
$= x$

$x^{5/5} = x^1 \rightarrow$

Recall:
$a^m a^n = a^{m+n}$

Example 4

Find each number on your calculator. (Round to the nearest 100[th]).

$\sqrt[7]{-14}$

$-\dfrac{1}{\sqrt[4]{9}}$

$\sqrt[7]{-14} \approx -1.46$

$-\dfrac{1}{\sqrt[4]{9}} \approx -.58$

Example 5

Find each number *without the use of your calculator.*

$9^{-1/2}$

$$9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

$125^{2/3}$

$$125^{2/3} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$$

$20^{1/2} \cdot 5^{1/2}$

$$20^{1/2}5^{1/2} = (20 \cdot 5)^{1/2} = 100^{1/2} = \sqrt{100} = 10$$

$12^{1/2} \cdot 12^{-5/2}$

$$12^{1/2} \cdot 12^{-5/2} = 12^{\frac{1}{2} + (-\frac{5}{2})} = 12^{-2} = \frac{1}{12^2} = \frac{1}{144}$$

$-8^{4/3}$

$$-8^{4/3} = -\left(\sqrt[3]{8}\right)^4 = -2^4 = -16$$

$-4^{-1/2}$

$$-4^{-1/2} = -\frac{1}{4^{1/2}} = -\frac{1}{\sqrt{4}} = -\frac{1}{2}$$

Example 6

Find each function value without the use of your calculator.

Find $f(-17)$ where $f(x)=(8-x)^{1/2}$.

$$f(-17) = (8-(-17))^{1/2}$$
$$= 25^{1/2}$$
$$= \sqrt{25}$$
$$= 5$$

Find $g(64)$ where $g(t)=-t^{-4/3}$

$$g(64) = -64^{-4/3}$$
$$= -\frac{1}{64^{4/3}}$$
$$= -\frac{1}{(\sqrt[3]{64})^4}$$
$$= -\frac{1}{4^4}$$
$$= -\frac{1}{256}$$

Find $k(-8)$ where $k(x)=(x+7)^{1/3}+x^{5/3}$

$$k(-8) = (-8+7)^{1/3}+(-8)^{5/3}$$
$$= (-1)^{1/3} + (-8)^{5/3}$$
$$= \sqrt[3]{-1} + (\sqrt[3]{-8})^5$$
$$= -1 + (-2)^5$$
$$= -1 + (-32)$$
$$= -33$$

$3+3=2(3)$

4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144

Combining Radical Expressions

Combining radical expressions is pretty much the same thing as combining like terms. You first need to completely simplify each radical expression. Then, if two or more terms have *exactly* the same radical factors, they can be added or subtracted into a single radical expression. If two terms have either a different index and/or a different radicand they cannot be combined.

Examples

Completely simplify each radical expression.

$\sqrt{8}+\sqrt{2}$

$$\begin{aligned}\sqrt{8}+\sqrt{2} &= \sqrt{4\cdot 2}+\sqrt{2}\\ &= \sqrt{4}\sqrt{2}+\sqrt{2}\\ &= 2\sqrt{2}+\sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

$\sqrt{12}+\sqrt{2}$

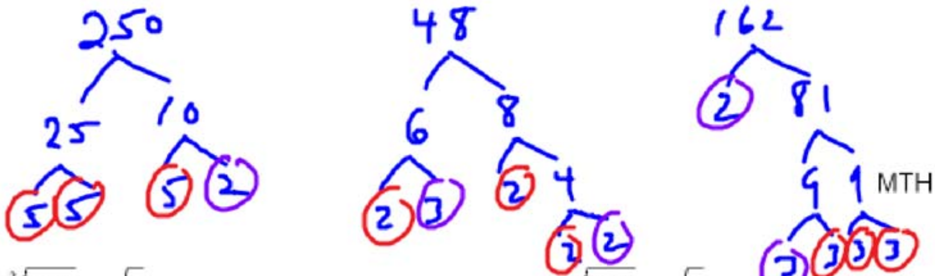
$$\begin{aligned}\sqrt{12}+\sqrt{2} &= \sqrt{4\cdot 3}+\sqrt{2}\\ &= \sqrt{4}\sqrt{3}+\sqrt{2}\\ &= 2\sqrt{3}+\sqrt{2}\end{aligned}$$

$4+\sqrt{50}$

$$\begin{aligned}4+\sqrt{50} &= 4+\sqrt{25\cdot 2}\\ &= 4+\sqrt{25}\sqrt{2}\\ &= 4+5\sqrt{2}\end{aligned}$$

$4\sqrt{5}+\sqrt{50}$

$$\begin{aligned}4\sqrt{5}+\sqrt{50} &= 4\sqrt{5}+\sqrt{25\cdot 2}\\ &= 4\sqrt{5}+\sqrt{25}\sqrt{2}\\ &= 4\sqrt{5}+5\sqrt{2}\end{aligned}$$



$\sqrt[3]{250}-\sqrt{5}$

$$\sqrt[3]{250}-\sqrt{5}=\sqrt[3]{125\cdot 2}-\sqrt{5}=\sqrt[3]{125}\sqrt[3]{2}-\sqrt{5}=5\sqrt[3]{2}-\sqrt{5}$$

$\sqrt{250}-\sqrt{5}$

$$\sqrt{250}-\sqrt{5}=\sqrt{25\cdot 10}-\sqrt{5}=\sqrt{25}\sqrt{10}-\sqrt{5}=5\sqrt{10}-\sqrt{5}$$

$\sqrt{125}-\sqrt{5}$

$$\sqrt{125}-\sqrt{5}=\sqrt{25\cdot 5}-\sqrt{5}=5\sqrt{5}-\sqrt{5}=4\sqrt{5}$$

$\sqrt[3]{48}-5\sqrt[3]{162}$

$$\sqrt[3]{48}-5\sqrt[3]{162}=\sqrt[3]{8\cdot 6}-5\sqrt[3]{27\cdot 6}=\sqrt[3]{8}\sqrt[3]{6}-5\sqrt[3]{27}\sqrt[3]{6}=2\sqrt[3]{6}-5(3\sqrt[3]{6})=2\sqrt[3]{6}-15\sqrt[3]{6}=-13\sqrt[3]{6}$$

$\left(2+\sqrt{7}\right)\left(2-\sqrt{7}\right)$

$$(2+\sqrt{7})(2-\sqrt{7})=4-\cancel{2\sqrt{7}}+\cancel{2\sqrt{7}}-7=-3$$

$\left(2+\sqrt{7}\right)^2$

$$(2+\sqrt{7})^2=(2+\sqrt{7})(2+\sqrt{7})=4+2\sqrt{7}+2\sqrt{7}+7=11+4\sqrt{7}$$

$\left(3\sqrt{10}+\sqrt{5}\right)\left(4\sqrt{5}-5\sqrt{2}\right)$

$$\begin{aligned}
\left(3\sqrt{10}+\sqrt{5}\right)\left(4\sqrt{5}-5\sqrt{2}\right) &= (3)(4)\sqrt{10\cdot5}-(3)(5)\sqrt{10\cdot2}+4\sqrt{5\cdot5}-5\sqrt{5\cdot2}\\
&= 12\sqrt{50}-15\sqrt{20}+4\sqrt{25}-5\sqrt{10}\\
&= 12\sqrt{25\cdot2}-15\sqrt{4\cdot5}+4(5)-5\sqrt{10}\\
&= 12(5\sqrt{2})-15(2\sqrt{5})+20-5\sqrt{10}\\
&= 60\sqrt{2}-30\sqrt{5}+20-5\sqrt{10}
\end{aligned}$$

$\left(\sqrt{6}-\sqrt{15}\right)^2$

$$\begin{aligned}
\left(\sqrt{6}-\sqrt{15}\right)^2 &= \left(\sqrt{6}-\sqrt{15}\right)\left(\sqrt{6}-\sqrt{15}\right)\\
&= 6-\sqrt{90}-\sqrt{90}+15\\
&= 6-\sqrt{9\cdot10}-\sqrt{9\cdot10}+15\\
&= 6-3\sqrt{10}-3\sqrt{10}+15\\
&= 21-6\sqrt{10}
\end{aligned}$$

$\left(\sqrt{6}-\sqrt{15}\right)\left(\sqrt{6}+\sqrt{15}\right)$

$$\begin{aligned}
\left(\sqrt{6}-\sqrt{15}\right)\left(\sqrt{6}+\sqrt{15}\right) &= 6+\sqrt{90}-\sqrt{90}-15\\
&= -9
\end{aligned}$$

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

Rationalizing binomial square root expressions

We frequently want to clear square roots out of a denominator - this process is called *rationalizing the denominator*. We sometimes (albeit rarely) even want to rationalize the numerator.

Expressions with two terms are called *binomials* and the pair of binomials $a+b$ and $a-b$ are called *conjugates*. Conjugates can be useful when rationalizing a numerator or denominator because the OI adds to zero when you FOIL. That is:

$$(a+b)(a-b) = a^2 - b^2$$

Examples

Rationalize the denominator and simplify: $\dfrac{3}{\sqrt{2}-\sqrt{5}}$

$$\frac{3}{\sqrt{2}-\sqrt{5}} = \frac{3}{\sqrt{2}-\sqrt{5}}\cdot\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$$

$$= \frac{3(\sqrt{2}+\sqrt{5})}{2-5}$$

$$= \frac{3(\sqrt{2}+\sqrt{5})}{-3}$$

$$= \frac{3}{-3}\cdot\frac{\sqrt{2}+\sqrt{5}}{1}$$

$$= -1(\sqrt{2}+\sqrt{5})$$

$$= -\sqrt{2}-\sqrt{5}$$

Rationalize the denominator and simplify: $\dfrac{\sqrt{7}}{7+\sqrt{7}}$

$$\frac{\sqrt{7}}{7+\sqrt{7}} = \frac{\sqrt{7}}{7+\sqrt{7}}\cdot\frac{7-\sqrt{7}}{7-\sqrt{7}}$$

$$= \frac{\sqrt{7}(7-\sqrt{7})}{49-7}$$

$$= \frac{7\sqrt{7}-7}{42}$$

$$= \frac{7(\sqrt{7}-1)}{7\cdot 6}$$

$$= \frac{7}{7}\cdot\frac{\sqrt{7}-1}{6}$$

$$= \frac{\sqrt{7}-1}{6}$$

Rationalize the denominator and simplify: $\frac{6-\sqrt{10}}{6+\sqrt{10}}$

$$\frac{6-\sqrt{10}}{6+\sqrt{10}} = \frac{6-\sqrt{10}}{6+\sqrt{10}} \cdot \frac{6-\sqrt{10}}{6-\sqrt{10}}$$

$$= \frac{36-6\sqrt{10}-6\sqrt{10}+10}{36-10}$$

$$= \frac{46-12\sqrt{10}}{26}$$

$$= \frac{\cancel{2}(23-6\sqrt{10})}{\cancel{2}\cdot 13}$$

$$= \frac{23-6\sqrt{10}}{13}$$

Rationalize the denominator and simplify: $\frac{\sqrt{11}-8}{8-\sqrt{11}}$

$$\frac{\sqrt{11}-8}{8-\sqrt{11}} = \frac{\sqrt{11}-8}{8-\sqrt{11}} \cdot \frac{8+\sqrt{11}}{8+\sqrt{11}}$$

$$= \frac{8\sqrt{11}+11-64-8\sqrt{11}}{64-11}$$

$$= \frac{-53}{-53}$$

$$= 1$$